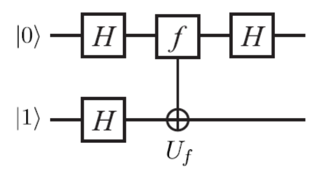
In this chapter, we introduce some simple algorithms. This demonstrate how one can use the quantum properties to solve certain problems efficiently. It should be emphasized that formulating the "right" questions to use quantum properties are important.

Simple quantum algorithms

5.1 Deutsch Algorithm

Let $f : \{0, 1\} \to \{0, 1\}$.
Decide whether $f(0) = f(1)$ or $f(0) \neq f(1)$
using one U_f evaluation.



Step 1 $|\psi_0\rangle = (H \otimes H)|01\rangle = (1/2)(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$.

Stpe 2 Let $U_f : |x, y\rangle \mapsto |x, y \oplus f(x)\rangle$. Then

$|\psi_1\rangle = U_f|\psi_0\rangle$

$= (1/2)(|0, f(0)\rangle - |0, 1 \oplus f(0)\rangle + |1, f(1)\rangle - |1, 1 \oplus f(1)\rangle)$

$= (1/2)(|0, f(0)\rangle - |0, \neg f(0)\rangle + |1, f(1)\rangle - |1, \neg f(1)\rangle)$.

Step 3 $|\psi_2\rangle = (H \otimes I_2)|\psi_1\rangle$

$= \gamma[(|0\rangle + |1\rangle)(|f(0)\rangle - |\neg f(0)\rangle) + (|0\rangle - |1\rangle)(|f(1)\rangle - |\neg f(1)\rangle)]$.

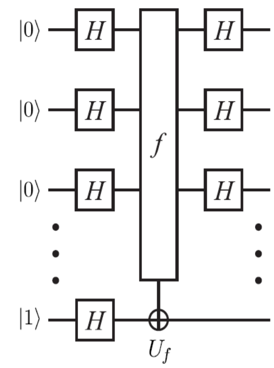
Step 4 Measure the first qubit of $|\psi_2\rangle$:

Case 1. If $f(0) = f(1)$, then $|\psi_2\rangle = |0\rangle(|f(0)\rangle - |\neg f(0)\rangle)$ and we get the measurement ....
Case 2. If $f(0) \neq f(1)$, then $|\psi_2\rangle = |1\rangle(|f(0)\rangle - |\neg f(0)\rangle)$ and we get the measurement ....

5.2.1 Deutsch-Jozsa Algorithm

Let $S_n = \{0, 1, \ldots, 2^n - 1\}$ and $f : S_n \to \{0, 1\}$.

We want to decide whether f is constant or balanced.



Step 0 $|\psi_0\rangle = |0\rangle^{\otimes n}|1\rangle$

Step 1 $|\psi_1\rangle = W_{n+1}|\psi_0\rangle = \gamma(\sum_x |x\rangle)(|0\rangle - |1\rangle)$.

Step 2 Let $U_f : |x\rangle|c\rangle \mapsto |x\rangle|c \oplus f(x)\rangle$ and set

$|\psi_2\rangle = U_f|\psi_1\rangle$

$= \gamma \sum_x (|x\rangle(|0\rangle - |1\rangle) \oplus f(x)\rangle)$

$= \gamma \sum_x |x\rangle(-1)^{f(x)}(|0\rangle - |1\rangle)$

(as $|c\rangle = |0\rangle - |1\rangle$ changes to $\pm|c\rangle$ depending on $f(x) = |0\rangle$ or $1\rangle$)

$= \gamma \sum_x (-1)^{f(x)}|x\rangle(|0\rangle - |1\rangle)$

Step 3 $|\psi_3\rangle = (W_n \otimes I_2)|\psi_2\rangle = \gamma \left(\sum_{x,y} (-1)^{f(x)} (-1)^{x \cdot y} |y\rangle \right) (|0\rangle - |1\rangle)$.

Note that $W_1|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $W_1|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

so that $\gamma W_1(|0\rangle + |1\rangle) = \sum_{x,y \in \{0,1\}} (-1)^{xy} |y\rangle$. Then

$W_2(\sum_{x_1 x_2} |x_1 x_2\rangle)$

$= (\sum_{x_1,y_1} (-1)^{x_1 y_1} |y_1\rangle)(\sum_{x_2,y_2} (-1)^{x_2 y_2} |y_2\rangle)$

$= \sum_{(x_1,x_2),(y_1,y_2)} (-1)^{(x_1,x_2)\cdot(y_1,y_2)} |y_1 y_2\rangle$

$= \sum_{x,y} (-1)^{x \cdot y} |y\rangle$.

Here we are summing up the entries in each row.

Step 4 Measure the first n qubits.

Case 1. If f is constant, then $|\psi_3\rangle = \tilde{\gamma}|0\rangle^{\otimes n}(|0\rangle - |1\rangle)$.

Case 2. If f is balanced, then the probability of the measurement of the first n-qubits equal $|y\rangle = |0\cdots 0\rangle$ is proportional to $\boxed{\sum_x (-1)^{f(x)}(-1)^{x\cdot 0} = \sum_x (-1)^{f(x)} = 0}$ because half of the $f(x)$ values are 0 and the rest are 1.

A closed look for a 2-qubit example

$f(0,0)=0, f(0,1)=1, f(1,0)=1, f(1,1)=0.$

$$U_f(W_2|00\rangle|1\rangle) = \gamma U_f \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\} \left(\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \right)$$

$$= \gamma \left\{ \begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\} \left(\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix} \right).$$

Applying $H \otimes H \otimes I_2$ to the vector, we get

$$\tilde{\gamma} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \otimes (|0\rangle - |1\rangle).$$

Then the measurement of the first two qubits yields $|00\rangle$ when all $(-1)^{f(x)}$ are equal for all $|x\rangle$.

Else, it has the form $\sum_y c_y|y\rangle$, where $c_y = \sum_x (-1)^{f(x)}(-1)^{x\cdot y}$.

Here note that:

$$W_2(\textstyle\sum_{x_1x_2}|x_1x_2\rangle) = (\sum_{x_1,y_1}(-1)^{x_1y_1}|y_1\rangle)(\sum_{x_2,y_2}(-1)^{x_2y_2}|y_2\rangle)$$
$$= \textstyle\sum_{(x_1,x_2),(y_1,y_2)}(-1)^{(x_1,x_2)\cdot(y_1,y_2)}|y_1y_2\rangle.$$

5.2.2 Bernstein-Vazirani algorithm

In the above example, $f(x_1,x_2) = (1,1)\cdot(x_1,x_2)$.

Then the resulting measurement of the first two qubits of $|\psi_3\rangle$ yields $|11\rangle$.

In general, let $f(x) = c \cdot x = (c_{n-1},\dots,c_0)\cdot(x_{n-1},\dots,x_0)$.

Apply the Deutsch-Jozsa algorithm to get:

$$|\psi_3\rangle = \gamma \left(\sum_{x,y} (-1)^{c\cdot x}(-1)^{x\cdot y}|y\rangle \right)(|0\rangle - |1\rangle) = \gamma|c\rangle(|0\rangle - |1\rangle).$$

Measuring the first n-qubits will give $c = (c_{n-1},\dots,c_0)$.

5.3 Simon Algorithm

Let $f : \{0,1\}^n \mapsto \{0,1\}^n$ be such that $f(x \oplus p) = f(x)$ for some nonzero $p \in \{0,1\}^n$. A classical computer may need $2^{n-1}+1$ evaluations of f.

The following quantum algorithm finds p using n evaluations of f.

1. Set $|\psi_0\rangle = |0\rangle|0\rangle$ in $\mathbf{C}^N \otimes \mathbf{C}^N$ with $N = 2^n$.

   Use the Walsh-Hadamard transformation W_n to get $|\psi_1\rangle = (W_n \otimes I)|\psi_0\rangle = \eta \sum_{x=0}^{2^n-1} |x\rangle|0\rangle$, where $\eta = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{2^n}}$.

2. Use U_f and n controlled-NOT gates with control qubits $f_1(x), \ldots, f_n(x)$ to get

   $|\psi_2\rangle = \eta \sum_x |x\rangle|f(x)\rangle$.

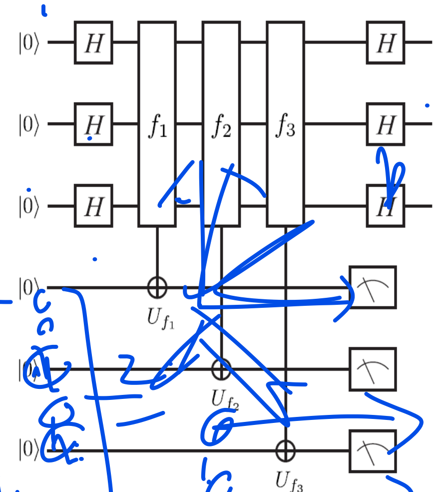
3. Apply measurement to the second state to get

   $|\psi_3\rangle = \frac{1}{\sqrt{2}}(|x_0\rangle + |x_0 + p\rangle)|f(x_0)\rangle$.

4. Apply $W_n \otimes I$ again to get

   $|\psi_4\rangle = \eta \sum_y (-1)^{x_0 \cdot y}[1 + (-1)^{p \cdot y}]|y\rangle|f(x_0)\rangle$

   $= \eta\sqrt{2} \sum_{p \cdot y = 0} (-1)^{x_0 \cdot y}|y\rangle|f(x_0)\rangle$.



5. Measure the first state to get $|y\rangle$ such that $p \cdot y = 0$.

   Here we use the fact that the only states $|y\rangle$ with positive probability in the sum are those satisfying $p \cdot y = 0$.

   Thus, a measurement will always yield such a vector $y_1 = (y_{11} \cdots y_{1n})$.

   Repeat this to get linearly independent $y_1, \ldots, y_{n-1}$ such that $p \cdot y_j = 0$ for all j, i.e., we have a linear system $(y_{ij})(p_0, \ldots, p_{n-1})^t = (0, \ldots, 0)^t$.

   We need to do it in $O(n)$ attempts with a good probability. Then solve for p.