

6.1 Quantum Integral Transform

Let  $S_n = \{0, \dots, N-1\}$  with  $N = 2^n$  and let  $K$  be an  $N \times N$  complex matrix with entries  $K(i, j)$  with  $i, j \in S_n$ . Then  $K$  is a QIT transform converting  $f = (f(0), \dots, f(N-1))^t$  to

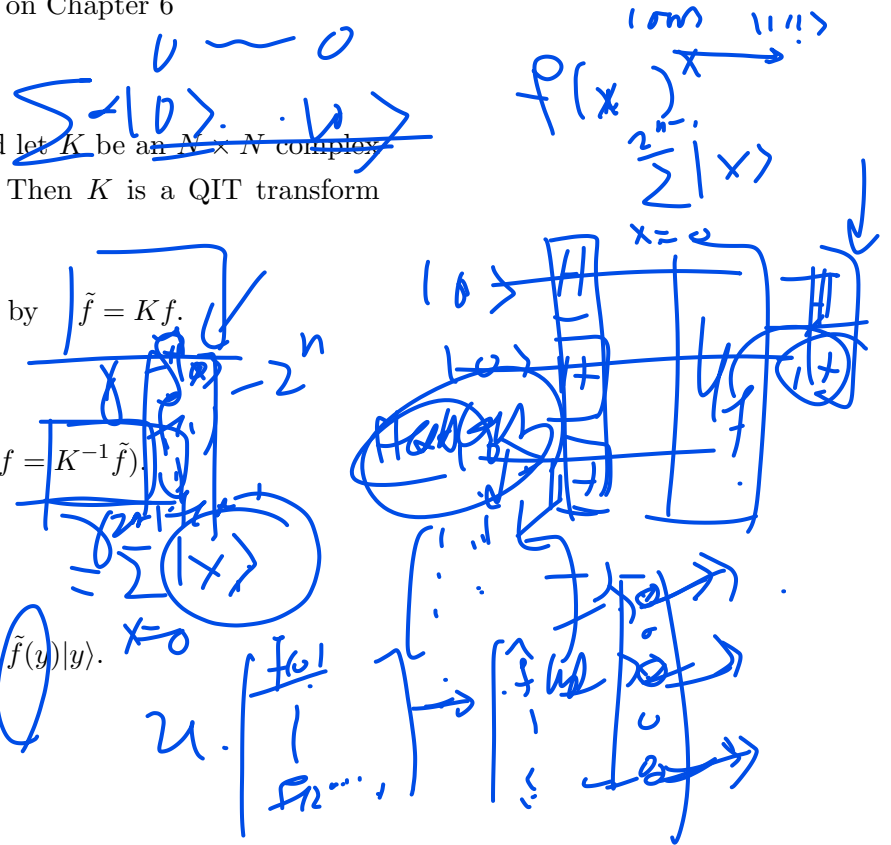
$$\tilde{f} = (\tilde{f}(0), \dots, \tilde{f}(N-1))^t \quad \text{by} \quad |\tilde{f} = Kf.$$

If  $K$  is unitary (invertible) then

$$f = K^\dagger \tilde{f} \quad (\text{respectively, } f = K^{-1} \tilde{f})$$

**Proposition** If  $U|x\rangle = K|y\rangle$ , then

$$U \left[ \sum_{x=0}^{2^n-1} f(x)|x\rangle \right] = \sum_{y=0}^{2^n-1} \tilde{f}(y)|y\rangle.$$



## 6.2 Quantum Fourier Transform

Suppose  $N = 2^n$ ,  $w = e^{2\pi i/N}$ ,  $\sqrt{N}$  and

$K = K(x, y)$  with  $K(x, y) = (w_n^{-xy})$ .

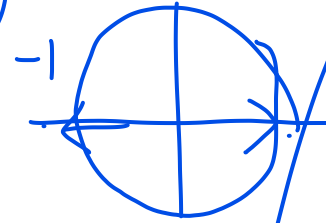
Then  $\tilde{f} = Kf$  is a commonly used QFT.

**Example** When  $n = 1, 2$ .

$n=1$   $N=2$

$w_4^j$   $e^{\frac{2\pi i}{2}} = e^{\pi i}$

$$K^2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} = H$$



$i^{-3}$

$$\begin{pmatrix} +i \\ -i \end{pmatrix}^{-2} \begin{pmatrix} -1 \\ 1 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} i \\ -i \end{pmatrix}^{-1}$$

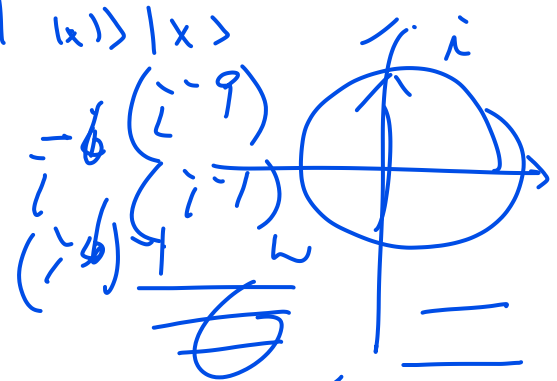
$(i^{-3})^{-1}$

$n=2$

$N=4$

$$\frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$$

$w_2 = i$



### 6.3 Application of QFT to period finding

This is an essential component in the Shor's algorithm.

For a periodic function,  $f : S_n \rightarrow S_n$ , where  $S_n = \mathbb{Z}_2^n$ , we want to detect  $P \in S_n$  such that

$$f(x) = f(x + P) \quad \text{for all } x \in S_n.$$

**Example** Let  $n = 3, P = 2; f(0) = f(2) = f(4) = f(6) = a,$   
 $f(1) = f(3) = f(5) = f(7) = b.$

Step 1. Prepare  $|\Psi_0\rangle = |0\rangle|0\rangle \in S_3 \otimes S_3.$

Step 2. Apply  $W_3 \otimes I_8$  to  $|\Psi_0\rangle$  and the oracle  $U_f$  to get  $|\Psi\rangle = \gamma \sum_x |x\rangle|f(x)\rangle.$

Step 3. Apply  $F = [e^{-2\pi ixy/8}] \otimes I_n$  to  $|\Psi\rangle$  to get

$$\begin{aligned} |\Psi'\rangle &= \gamma \sum_{x,y} e^{-2\pi ixy/8} |y, f(x)\rangle \\ &= \gamma \left( |0\rangle[|f(0)\rangle + |f(1)\rangle + \dots + |f(7)\rangle] \quad (y=0) \right. \\ &\quad + \gamma |1\rangle[|f(0)\rangle + e^{2\pi i/8}|f(1)\rangle + \dots + e^{-2\pi i7/8}|f(7)\rangle] \quad (y=1) \\ &\quad + \dots \\ &\quad \left. + \gamma |7\rangle[|f(0)\rangle + e^{-14\pi i/8}|f(1)\rangle + \dots + e^{-14\pi i7/8}|f(7)\rangle] \quad (y=7) \right) \\ &= \frac{1}{2} (|0, a\rangle + |0, b\rangle + |4, a\rangle + e^{-i\pi}|4, b\rangle). \end{aligned}$$

Step 4. Measurement of the first register gives 0, 4. So the period is 2.

**Remark** Table 6.2 is not accurate.

TABLE 6.3

Coefficient of a vector  $|y\rangle|f(x)\rangle$  in the state  $|\Psi'\rangle$  in which  $f(0) = f(2) = f(4) = f(6) = a$  and  $f(1) = f(3) = f(5) = f(7) = b$ . The amplitude of all the non-vanishing coefficients is  $1/2$ .

$ b\rangle$	$\rightarrow$	0	0	0	$\leftarrow$	0	0	0	
$ a\rangle$	$\rightarrow$	0	0	0	$\rightarrow$	0	0	0	
		$ 0\rangle$	$ 1\rangle$	$ 2\rangle$	$ 3\rangle$	$ 4\rangle$	$ 5\rangle$	$ 6\rangle$	$ 7\rangle$

**Remark** In general, the observed value of the first register is one of

$$\frac{1}{P}k \cdot 2^n, \quad k = 0, 1, \dots, P-1.$$

$f: \{0, 1, \dots, 2^n - 1\}$

$\rightarrow \{0, 1, \dots, 2^n - 1\}$

$|000\rangle |000\rangle$

$\sum_x |x\rangle |0\rangle$

$\sum_x |x\rangle |f(x)\rangle$

$x=0$

$|0\rangle |0\rangle$

$|4\rangle |a\rangle$

$|4\rangle |b\rangle$

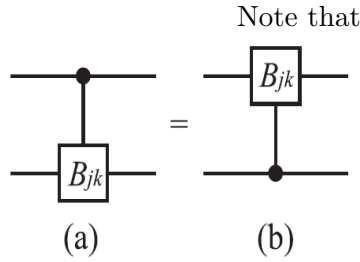
### 6.4 Implementation of QFT

When  $n = 1$ .  $U_{QFT_1} = W_1$ .

When  $n \geq 2$ , we need the controlled

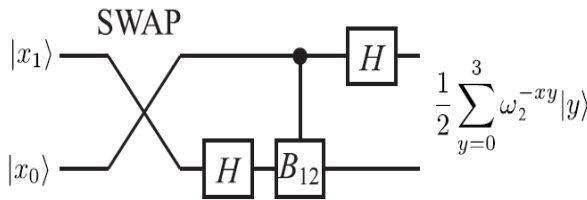
$$B_{jk} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i2\pi/(k-j+1)} \end{pmatrix} \text{ for } k \geq j$$

and the SWAP gate to implement  $U_{QFT_2}$ .



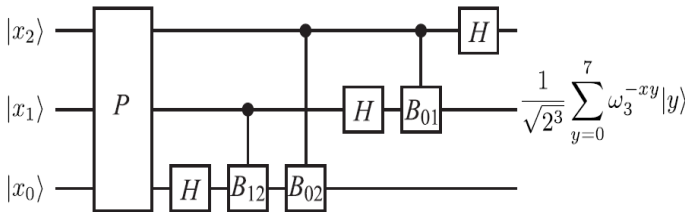
**PROPOSITION 6.3** The  $n = 2$  QFT gate is implemented as

$$U_{QFT_2} = (U_H \otimes I)U_{12}(I \otimes U_H)U_{SWAP}$$

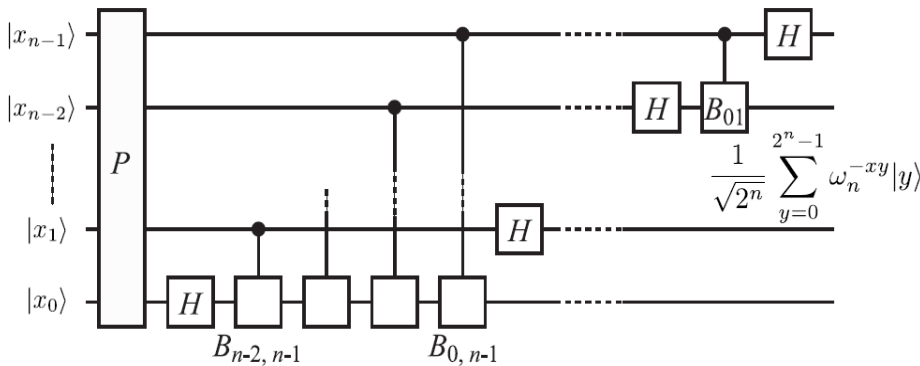


**Note**  $U_{QFT_n} = U_{QFT_n}^t$ . So, ...

When  $n = 3$ , we have the following.



When  $n \geq 3$ , we have the following.



**Proposition** QFT can be implemented using  $O(n^2)$  elementary gates.

## 6.5 Walsh-Hadamard Transform

The kernel  $W_n = ((-1)^{x \cdot y})$  defines the discrete integral transform

$$\tilde{f}(y) = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} (-1)^{x \cdot y} f(x).$$

## 6.6 Selective Phase Rotation Transform

The kernel  $\text{diag}(\theta_0, \dots, \theta_{N-1})$  defines the transform

$$\tilde{f}(y) = \sum e^{i\theta_x} \delta_{xy} f(x) = e^{i\theta_y} f(y).$$

Note that

$$K_1 = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix}, \quad K_2 = \begin{pmatrix} e^{i\theta_0} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix}.$$

Here

$$K_2 = A_0 A_1,$$

with

$$A_0 = \begin{pmatrix} e^{i\theta_0} & 0 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & e^{i\theta_2} & 0 \\ 0 & 0 & 0 & e^{i\theta_3} \end{pmatrix},$$

$$A_0 = |0\rangle\langle 0| \otimes U_0 + |1\rangle\langle 1| \otimes I, \quad U_0 = \begin{pmatrix} e^{i\theta_0} & 0 \\ 0 & e^{i\theta_1} \end{pmatrix},$$

$$A_1 = |0\rangle\langle 0| \otimes I + |1\rangle\langle 1| \otimes U_1, \quad U_1 = \begin{pmatrix} e^{i\theta_2} & 0 \\ 0 & e^{i\theta_3} \end{pmatrix}.$$

**Remark** One can actually write

$$K_2 = (I \otimes K_1) \hat{A}_2 \quad \text{with} \quad \hat{A}_1 = \text{diag}(1, 1, e^{i(\theta_2 - \theta_0)}, e^{i(\theta_2 - \theta_1)}).$$