$\Phi: M_n \to M_m$

$\Phi(A) =$

Theorem For every quantum operation on an open system $\Phi : M_n \to M_m$ there exist $r \in \mathbb{N}$ and $F_1, \dots, F_r \in M_{m,n}$ such that $\sum_{j=1}^r F_j^\dagger F_j = I_n$ and

$m \times n$

$$\Phi(A) = \sum_{j=1}^r F_j A F_j^\dagger \quad \text{for all } A \in M_n.$$

$m \times m$

This is called the operator sum representation of the quantum operation. The matrices $F_1, \dots, F_r$ are called the Kraus operators of the operations.

Proof. Suppose $\Phi : M_n \to M_m$ is an quantum operation.

We may assume that $\Phi(\rho)$ is the partial trace of

$$U(\sigma \otimes \rho)U^\dagger \in M_{nk} \quad \text{with } nk = mr.$$

Here U may depends on t. By purification, we may assume that $\sigma = E_{11}$ so that

$$U(\sigma \otimes \rho)U^\dagger = U \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} U^\dagger = \begin{pmatrix} F_1 \\ \vdots \\ F_r \end{pmatrix} \rho (F_1^* | \cdots | F_r^*)$$

with diagonal blocks $F_1 \rho F_1^\dagger, \dots, F_r \rho F_r^\dagger$ so that

$$\mathrm{tr}_1(U(\sigma \otimes \rho)U^\dagger = \sum_{j=1}^r F_j \rho F_j^\dagger.$$

Here $(F_1^\dagger, \dots, F_r^\dagger)$ are the first n rows of $U^\dagger$.

Thus, $\sum_{j=1}^r F_j^\dagger F_j = I_n$.

Remark quantum channels are trace preserving completely positive linear maps.

$\Phi(\rho) = \sigma$

Example Let $U_1, \ldots, U_r \in \mathrm{U}(n)$ and $p_1, \ldots, p_r$ be positive numbers summing up to 1. Then $\Phi : M_n \to M_n$ defined by

$$\Phi(A) = \sum_{j=1}^{r} p_j U_j A U_j^\dagger \quad \text{for all } A \in M_n$$

$\Phi(A) = B$

is a quantum channel known as the **random unitary channel** or **mixed unitary channel**.

$\Phi \otimes I_2 : M_{2n} \to M_{2n}$

$\Phi \otimes I_2\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\right) \geq 0$ psd

Quantum channels and Measurements

When a quantum state ρ is transmitted through a quantum channel, it will interact with the external environment. So, we may regard the transmission as a process of letting the quantum state going through a quantum operation of an open system, and assume the received state has the form

$$\hat{\rho} = \sum_{j=1}^{r} F_j \rho F_j^\dagger.$$

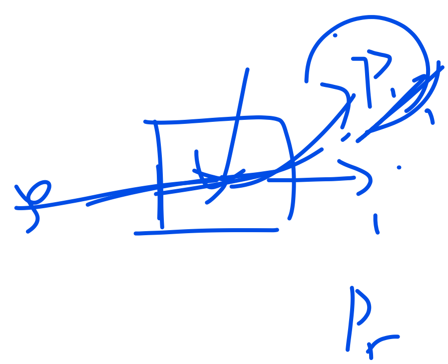
Here $F_1, \ldots, F_r$ are the Kraus operators caused by the influence of the environment on ρ. In this context, $F_1, \ldots, F_r$ are known a the **error operators**.

Positive Operator-Valued Measure (POVM)

- Eigenprojections of A.

  Quantum measurements can be viewed as quantum operations on open systems. As mentioned before a Hermitian matrix $A = \sum_{j=1}^{n} \lambda_j |\lambda_j\rangle\langle\lambda_j|$ is associated with an observable. If a state $\rho \in D_n$ goes through the measurement process corresponding to A, the state ρ will "collapse" to one of the pure states $|\lambda_j\rangle\langle\lambda_j|$ with a probability $\mathrm{tr}\,(A\rho)$.

- Projective measurement.

  In general, if $A = \sum_{j=1}^{s} \lambda_j P_j$, where P_j is the projection operator corresponding to the eigenvalue λ_j for the distinct eigenvalues $\lambda_1, \ldots, \lambda_s$ of A. In such a case, the **projective measurement** of ρ under the measurement associated with A is the quantum operation

  $$\boxed{\rho \mapsto \sum_j P_j \rho P_j,} \qquad F_j^\dagger = F_j$$

  where $p_j = \mathrm{tr}\,(P_j \rho P_j) = \mathrm{tr}\,(\rho P_j)$ and the set $\{P_1, \ldots, P_r\}$ satisfies the completeness relation $\sum_j P_j P_j^\dagger = \sum_j P_j = I$.

- POVM. for any positive semidefinite matrices $Q_1, \ldots, Q_r \in M_n$ such that $Q_1 + \cdots + Q_r = I_n$, there are $M_1, \ldots, M_r \in M_n$ such that $M_j^\dagger M_j = Q_j$. The measurement operators are then associated with the quantum operation

$$\rho \mapsto \sum_{j=1}^{r} M_j \rho M_j^\dagger$$

so that ρ will change to the quantum state $\frac{1}{p_j} M_j \rho M_j^\dagger$ with a probability $p_j = \mathrm{tr}\,(M_j \rho M_j^\dagger) = \mathrm{tr}\,(\rho Q_j)$. The set $\{Q_1, \ldots, Q_r\} = \{M_1^\dagger M_1, \ldots, M_r^\dagger M_r\}$ is known as the **positive operator-valued measure (POVM)**.

ρ

$p_1 + \cdots + p_r = 1$

$\Phi : M_n \to M_m$

$\Phi(\rho) = \sum_j F_j \rho F_j^\dagger = \boxed{\sigma}$

Example Suppose Bob will be given a quantum state chosen from the linearly independent set of unit vectors $\{|\psi_1\rangle, \ldots, |\psi_m\rangle\}$, which may not be orthonormal. He can construct the following POVM $\{Q_1, \ldots, Q_{m+1}\}$ such that he will know for sure that $|\psi_j$ is sent to him if the measurement of the received state yields Q_j if $Q_j = |\phi_j\rangle\langle\phi_j|/m$, where $\langle\phi_j|\phi_j\rangle = 1$ and $\langle\phi_j|\psi_i\rangle = 0$ for all $i \neq j$ for $j = 1, \ldots, m$ and $Q_{m+1} = I - \sum_{j=1}^{m} Q_j$. Evidently, a measurement of $|\psi_j\rangle\langle\psi_j|$ will yield Q_j or Q_{m+1}.

Fidelity

Definition Let $\rho_1, \rho_2 \in D_n$. Then the fidelity is defined by

$$F(\rho_1,\rho_2) = \left\{ \mathrm{tr}\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}} \right) \right\}^2 .$$

$\sqrt{\sum \lambda_i |x_i\rangle\langle x_i|}$

Here, $\sqrt{\rho_1}$ is the positive semi-definite square root of ρ_1, and $\sqrt{\rho_1}\rho_2\sqrt{\rho_1}$ is positive semi-definite so that we can take its positive semi-definite square root.

Theorem Let $\rho_1, \rho_2 \in \mathcal{S}(\mathcal{H})$. If $\rho_1^{1/2}\rho_2^{1/2}$ has singular values $s_1 \geq \cdots \geq s_n$, then

$\rho^{\frac{1}{2}} = \sqrt{\rho}$

$$F(\rho_1,\rho_2) = F(\rho_2,\rho_1) = \left[\sum_{j=1}^{n} s_j \right]^2 ,$$

and the following conditions hold.

(1) For any unitary U, $F(U\rho_1U^\dagger, U\rho_2U^\dagger) = F(\rho_1,\rho_2)$.

$\langle \rho_1, \rho_2 \rangle$

(2) If ρ_1 or ρ_2 is a pure state, then $F(\rho_1,\rho_2) = \mathrm{tr}\,(\rho_1\rho_2)$.

$= \mathrm{tr}(\rho_1\rho_2)$

(3) We have

$$F(\rho_1,\rho_2) \in [0,1].$$

The equality $F(\rho_1,\rho_2) = 1$ holds if and only if $\rho_1 = \rho_2$. The equality $F(\rho_1,\rho_2) = 0$ holds if and only if $\mathrm{tr}\,(\rho_1\rho_2) = 0$, equivalently, $\sigma_1^r\sigma_2^s = 0$ for any positive numbers r, s.

$\Longleftrightarrow \rho_1^r\rho_2^s = 0$

Other numerical functions on a mixed states

- The trace distance: $0 \le \|\rho - \sigma\|_{\mathrm{tr}} \le 2$ is the sum of the singular values of $\rho - \sigma$.

- The relative entropy of two quantum states $\rho, \sigma \in D_n$ defined by
$$S(\rho\|\sigma) = -\operatorname{tr}\rho \ln \sigma + \operatorname{tr}\rho \ln \rho = \operatorname{tr}\rho(\rho - \sigma)$$
is another measure of the difference between the two quantum states. If there is $|v\rangle \in \mathbb{C}^n$ such that $\sigma|v\rangle = 0$ and $\langle v|\rho|v\rangle \neq 0$, then $S(\rho\|\sigma) = \infty$.

- The von Neumann entropy of a density matrix is defined as
$$\boxed{S(\rho) = -\operatorname{tr}(\rho \ln \rho),}$$
where ln is the natural log function.

$$\rho = \sum_{i=1}^{n} \lambda_i |\lambda_i\rangle\langle\lambda_i|$$

$$= -\sum_{i=1}^{n} \lambda_i \ln \lambda_i$$

$$\rho = \begin{bmatrix} 1 & \\ & 0 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & \\ & 1 \end{bmatrix}$$

$$S_1 = S_2 = 1$$

$$U\rho U^\dagger = \begin{bmatrix} D_1 & 0 \\ 0 & 0 \end{bmatrix}$$

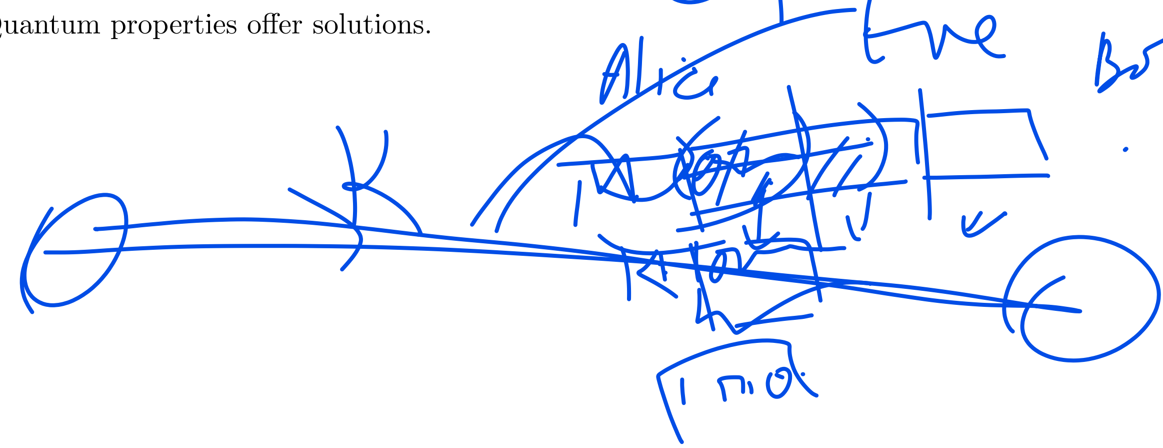
$$U\sigma U^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & D_2 \end{bmatrix}$$

Quantum Key distribution

Information are encrypted as $0-1$ sequences. Alice and Bob use a private key K, a (long) 0-1 sequence to encrypt their messages so that M is encrypted as $\tilde{M} = M + K$ and decrypted by $\tilde{M} + K$.

Problem How to exchanged the private key securely?

Quantum properties offer solutions.

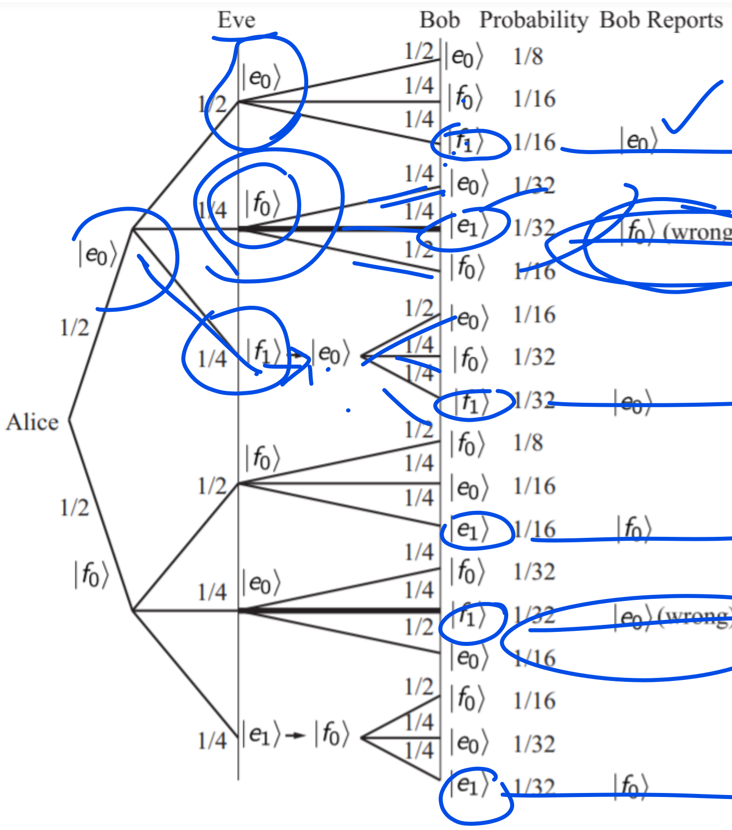


BB84 (Bennett and Brassard, 1984)

- Each of Alice and Bob use two bases for photon states: $B_1 = \{|e_0\rangle, |e_1\rangle\}$, $B_2 = \{|f_0\rangle, |f_1\rangle\}$.
- Alice sends Bob $4N$ photons, each prepared in one of the two bases randomly.
- Bob measures the received photons, each with one of the two bases randomly.
- Then they exchange notes and identify the photons that were sent and measured using the same bases. There should be roughly $2N$ such photons.
- They will use N of them to detect whether there is an eavesdropper, Eve, tampering their information.
- If Eve does intercept, the best things for her to do is to use B_1 and B_2 bases to measured the intercepted qubit, and then sends the measured qubit to Bob, using the same basis she obtains the measured result.
- Now, consider two cases. If Alice and Bob both used B_1, they should get a perfect match of information. However, if Eve has applied B_1 or B_2, about 1/2 of the times she would use B_2, and sent out the bit so that 1/2 of the times that Bob will get the measured results agree with the photon sent by Alice.
- The same holds if both Alice and Bob used $B2$. So, roughly 1/4 of the N-bits would disagree. Alice and Bob would deduce that someone has intercept the information if there is a huge discrepancy in the N-bits comparison, and should retry the process.

B92 Protocol

- Alice sends $8N$ photons to Bob using $|e_0\rangle$ for 0 and $|f_0\rangle$ for 1.
- Bob measures the received photons using B_1 or B_2 randomly.
- Suppose Alice sends $|e_0\rangle$. If Bob uses B_1, he will obtain $|e_0\rangle$; if Bob uses B_2, he will obtain $|f_0\rangle$ or $|f_1\rangle$. If he gets $|f_1\rangle$, he knows that Alice has sent $|e_0\rangle$.
- Suppose Alice send $|f_0\rangle$. If Bob uses B_2, he will obtain $|f_0\rangle$; if he uses B_1, he will obtain $|e_0\rangle$ or $|e_1\rangle$. If he gets $|e_1\rangle$, he knows that Alice has sent $f_0\rangle$.
- There are roughly $2N$ photons that Bob will know with certainty.
- He will use N of them to check the presence of Eve.
- If Eve indeed present, $1/16$ of the bits will fails to match.



8N

$\boxed{2N}$

N

$\frac{1}{16}$

There are **E91** and **BBM92** protocols using entangled pairs and Bell-Inequalities to check the presence of Eve.