Math 410 Quantum Computing C.K. Li Notes on Chapter 7

§7.1 Search for a single file

Let $f : S_n \to \{0, 1\}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x = z, \\ 0, & \text{if } x \neq z. \end{cases}$$

Step 1 Define the reflection R_f such that $R_f = I - 2|z\rangle\langle z|$. We have

$$R_f f = \sum_x f(x) R_f |x\rangle = \sum_x (-1)^{f(x)} |x\rangle.$$

Here, we simply have $(I - 2|z\rangle\langle z|)(\sum_{j=1}^N f(x)|x\rangle = ...)$.

Step 2 Construct

$$D = 2|\varphi_0\rangle\langle\varphi_0| - I_N = W_n(2|0\rangle\langle 0| - I_N)W_n = 2J_N/N - I_N,$$

where J_N is the matrix with all entries equal to 1.

If $|w\rangle = \begin{pmatrix} w_0 \\ \vdots \\ w_{N-1} \end{pmatrix}$, then

$$D|w\rangle = (2J_N/N - I)|w\rangle = 2\hat{w}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} - \begin{pmatrix} w_0 \\ \vdots \\ w_{N-1} \end{pmatrix}$$

with $\hat{w} = (\sum_j w_j)/N$.

Step 3 Construct $U_f = DR_f$ its action on $|\varphi_0\rangle = \sum_x |x\rangle$. Then

$$|\varphi_k\rangle = U_f^k|\varphi_0\rangle = a_k|z\rangle + b_k \sum_{x \neq z} |x\rangle$$

such that $a_0 = b_0 = 1/\sqrt{N}$. For $k \geq 1$ we have

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \frac{1}{N} \begin{pmatrix} N-2 & 2(N-1) \\ -2 & N-2 \end{pmatrix} \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix}.$$

Remark Note that

$$\begin{aligned} U_f|\varphi_{k-1}\rangle &= (2J_N/N - I)(I - 2|z\rangle\langle z|)|\varphi_{k-1}\rangle \\ &= (2J_N/N - I)(b_{k-1}, \ldots, b_{k-1}, -a_{k-1}, b_{k-1}, \ldots, b_{k-1})^t \\ &= (b_k, \ldots, b_k, a_k, b_k, \ldots, b_k)^t. \end{aligned}$$

Let $c_k = \sqrt{N-1}b_k$. If $(a_0, c_0) = (1, \sqrt{N-1})/\sqrt{N} = (\sin\theta, \cos\theta)$, then

$$\begin{pmatrix} a_k \\ c_k \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix} \begin{pmatrix} a_{k-1} \\ c_{k-1} \end{pmatrix} = \begin{pmatrix} \sin[(2k+1)\theta] \\ \cos[(2k+1)\theta] \end{pmatrix}.$$

Step 4 Maximize $P_{z,k}^2 = a_k^2$ by putting $(2k+1)\theta \approx \pi/2$. For large N we have $m = \lfloor \pi/4\theta \rfloor$ so that $m = O(\sqrt{N})$.

§7.2 Search for d files

Let $A \subseteq S_n$ have d elements, and $f : S_n \to \{0, 1\}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Step 1 Define the reflection R_f such that

$$R_f = I - 2 \sum_{z \in A} |z\rangle\langle z|.$$

Then for $|\varphi\rangle = \sum_{x=0}^{N-1} w_x |x\rangle$, $R_f(\varphi) = \sum_{x \notin A} w_x |x\rangle - \sum_{z \in A} w_z |z \in A\rangle$.

Step 2 Construct $D = -I + 2|\varphi_0\rangle\langle|\varphi_0|$ with $|\varphi_0\rangle = \sum_{x=0}^{N} |x\rangle/\sqrt{N}$.

Step 3 Construct $U_f = DR_f$ and its action on

$|\varphi\rangle = \sum_x w_x |x\rangle$ with $\sum_x |w_x|^2 = 1$.

Then $U_f^k|\varphi_0\rangle = a_k \sum_{z\in A} |z\rangle + b_k \sum_{x\notin A} |x\rangle$ such that $a_0 = b_0 = 1/\sqrt{N}$ and for $k \geq 1$ we have

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \frac{1}{N} \begin{pmatrix} N-2d & 2(N-d) \\ -2d & N-2d \end{pmatrix} \begin{pmatrix} a_{k-1} \\ b_{k-1} \end{pmatrix}$$

so that

$$\begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} (\sin[(2k+1)\theta])/\sqrt{d} \\ (\cos[(2k+1)\theta])/\sqrt{N-d} \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \begin{pmatrix} \sqrt{d/N} \\ \sqrt{1-d/N} \end{pmatrix}.$$

Grover's Search Algorithm