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Controllability of dynamical systems

Abstract. The paper contains systems descriptions and fundamental results concerning the solution of the most popular linear continuous-time control models with constant coefficients. First, different kinds of stability are discussed. Next fundamental definitions of controllability both for finite-dimensional and infinite-dimensional systems are recalled and necessary and sufficient conditions for different kinds of controllability are formulated.

Moreover, fundamental definitions of controllability both for finite-dimensional and infinite-dimensional control systems are presented and necessary and sufficient conditions for different kinds of controllability are given. Finally, concluding remarks and comments concerning possible extensions are presented.
Key words: linear systems, controllability, stability, distributed parameters systems.

1. Introduction. Controllability is one of the fundamental concepts in modern mathematical control theory. This is qualitative property of control systems and is of particular importance in control theory. Systematic study of controllability was started at the beginning of sixties in XX century and theory of controllability is based on the mathematical description of the dynamical system.

Many dynamical systems are such that the control does not affect the complete state of the dynamical system but only a part of it. On the other hand, very often in real industrial processes it is possible to observe only a certain part of the complete state of the dynamical system. Therefore, it is very important to determine whether or not control of the complete state of the dynamical system is possible. Roughly speaking, controllability generally means, that it is possible to steer dynamical system from an arbitrary initial state to an arbitrary final state using the set of admissible controls.

Controllability plays an essential role in the development of the modern mathematical control theory. There are important relationships between controllability, stability and stabilizability of linear control systems

[2], [3]. Controllability is also strongly connected with the theory of minimal realization of linear time-invariant control systems. Moreover, it should be pointed out that there exists a formal duality between the concepts of controllability and observability [2], [5].

In the literature there are many different definitions of controllability and stability which depend on the type of dynamical control system [1-10]. The main purpose of this paper is to present a compact review over the existing controllability results mainly for linear continuous-time, time-invariant control systems. It should be pointed out, that for linear control dynamical systems, the most popular controllability conditions have pure algebraic forms and hence are rather easily computable. These conditions require verification of the rank conditions for suitable defined constant controllability matrix.

This survey article is divided into sections and subsections and is organized as follows. Section 2 contains systems descriptions and fundamental results concerning the solution of the most popular and most frequently used linear continuous-time control models with constant coefficients. Section 3 presents fundamental definitions of controllability and the most frequently used sufficient and necessary conditions for different kinds of controllability. In Section 4 under suitable assumptions minimum energy control problem is analytically solved. Section 5 contains fundamental definition of controllability for linear infinite-dimensional dynamical systems and necessary and sufficient conditions for approximate controllability. Section 6 is devoted to a study of controllability for distributed parameters systems described by linear partial differential equations. Since the article should be limited to a reasonable size, it is impossible to give a full survey on the subject. In consequence, only selected fundamental results without proofs are presented. The wide treatment on controllability problems for different dynamical systems can be found in the monographs [1], [3], [5].

2. Mathematical model. In the theory of linear time-invariant dynamical control systems the most popular and the most frequently used mathematical model is given by the following differential state equation and algebraic output equations

$$x'(t) = Ax(t) + Bu(t) \tag{1}$$

$$y(t) = Cx(t) \tag{2}$$

where $x(t) \in R^n$ is a state vector, $u(t) \in R^m$ is an input vector, $y(t) \in R^p$ is an output vector, $t \geq 0$, A, B and C are real matrices of appropriate dimensions.

It is well known that for a given initial state $x(0) \in R^n$ and control $u(t) \in R^m$, $t \geq 0$, there exist unique solution $x(t; x(0), u) \in R^n$ of the differential state equation (1) of the following form

$$x(t;x(0),u) = e^{tA}x(0) + \int_0^t e^{(t-s)A}Bu(s)ds$$

Let P be an $n \times n$ constant nonsingular transformation matrix and let us define the equivalence transformation $z(t) = Px(t)$. Then the differential state equation (1) and output equation (2) becomes

$$z'(t) = Jz(t) + Gu(t) \tag{3}$$

$$y(t) = Hz(t) \tag{4}$$

where $J = PAP^{-1}$, $G = PB$ and $H = CP^{-1}$.

Dynamical systems (1), (2) and (3), (4) are said to be equivalent and many of their properties are invariant under the equivalence transformations.

Taking into account controllability concept, among different equivalence transformations special attention should be paid on the transformation, which leads to the so-called Jordan canonical form of dynamical system (1). In the case when the $n \times n$ dimensional matrix J is in Jordan canonical form, then the equation (3), (4) are said to be in a Jordan canonical form. Moreover, it should be stressed, that every dynamical system (1), (2) has an equivalent Jordan canonical form.

3. Controllability

3.1. *Fundamental results.* Now, let us recall the most popular and most frequently used fundamental definition of controllability for linear control systems with constant coefficients.

Definition 1. Dynamical system (1) is said to be controllable if for every initial condition $x(0)$ and every vector $x^1 \in R^n$, there exist a finite time t_1 and control $u(t) \in R^m$, $t \in [0, t_1]$, such that $x(t_1; x(0), u) = x^1$.

This definition requires only that any initial state $\mathbf{x}(0)$ can be steered to any final state x^1 at time t_1. However, the trajectory of the dynamical system (1) between 0 and t_1 is not specified. Furthermore, there is no constraints posed on the control vector $u(t)$ and the state vector $x(t)$.

In order to formulate easily computable algebraic controllability criteria let us introduce the so-called controllability matrix W, which is known as Kalman matrix and defined as follows.

$$W = [B|AB|A^2B|\ldots|A^kB|\ldots|A^{n-1}B].$$

It should be pointed out, that controllability matrix W is an $n \times nm$-dimensional constant matrix and depends only on system parameters.

Theorem 1. *Dynamical system* (1) *is controllable if and only if*

$$rank\ W = n.$$

Proof (a sketch). First of all, from Definition 1 and the form of solution of the state equation it follows, that dynamical system (1) is controllable if and only if for certain time $t = t_1$ the range of integral operator

$$\int_0^{t_1} e^{(t_1-s)A} Bu(s)ds$$

is the whole state space R^n. However, since $e^{t_1 A}$ is nonsingular for any t_1, then this is true if and only if symmetric $n \times n$ constant matrix

$$\int_0^{t_1} e^{-sA} BB^T e^{-sA^T} ds$$

is nonsingular. Taking into account Taylor series expansion of $e^{(-sA)}$ and the well known Cayley-Hamilton theorem we conclude, that dynamical system is controllable if and only if $rank\ W = n$.

Corollary 1. *Dynamical system* (1) *is controllable if and only if the* $n \times n$*-dimensional symmetric matrix* WW^T *is nonsingular.*

Since the controllability matrix W does not depend on time t_1, then from Theorem 1 and Corollary 1 it directly follows, that in fact controllability of dynamical system does not depend on the length of control interval. However, this statement is valid only for dynamical systems (1) without any constraints posed on the control vector $y(t)$ and the state vector $x(t)$.

Let us observe that in many cases in order to check controllability, it is not necessary to calculate whole controllability matrix W, but only a matrix with the same number of rows but with a smaller number of columns. It depends on the rank of the matrix B and the degree of the minimal polynomial for the matrix A, where the minimal polynomial is the polynomial of the lowest degree, which annihilates matrix A. This observation is based on the following Corollary.

Corollary 2. *Let rank* $B = r$, *and* q *is the degree of the minimal polynomial of the matrix* A. *Then dynamical system* (1) *is controllable if and only if*

$$rank\,[B|AB|A^2B|\dots|A^kB|\dots|A^{n-k}B] = n$$

where the integer $k \leq \min(n-r, q-1)$.

In the case when the eigenvalues of the matrix A, s_i, $i = 1, 2, 3, \dots, n$, are known, we can check controllability using necessary and sufficient condition given in the following Corollary, which is known as Hautus criterion.

Corollary 3. *Dynamical system* (1) *is controllable if and only if*

$$rank\,[s_iI - A|B] = n \text{ for all } s_i \in sp(A),\ i = 1, 2, 3, \dots, n.$$

Suppose that the dynamical system (1) is controllable. Then the dynamical system remains controllable after the equivalence transformation in the state space R^n. This is natural and intuitively clear because an equivalence transformation changes only the basis of the state space and does not change the properties of the dynamical system (1). Therefore, we have the following Corollary.

COROLLARY 4. *Controllability of the dynamical system* (1) *is invariant under any equivalence transformation in the state space* R^n.

Since controllability of dynamical system (1) is preserved under any equivalence transformation, then it is possible to obtain simpler controllability criteria by transforming the original differential state equation (1) into its special canonical form (3). For example, if we transform dynamical system (1) into Jordan canonical form, then controllability can be determined very easily, almost by inspection [5].

3.2. *Stabilizability.* It is well known that for linear dynamical system (1) there are certain relationships between controllability and stability. In order to explain these connections, let us introduce stability definitions. First we need the concept of equilibrium state.

DEFINITION 2. A state x^e of a dynamical system (1) is said to be an equilibrium state if and only if $x^e = x(t; x^e, 0)$ for all $t \geq 0$.

We see from this definition that if a trajectory reaches an equilibrium state and if no input is applied the trajectory will stay at the equilibrium state forever. Clearly, for linear dynamical systems the zero state is always an equilibrium state.

DEFINITION 3. An equilibrium state x^e is said to be stable if and only if for any positive ε, there exists a positive number $\delta(\varepsilon)$ such that inequality

$$\|x(0) - x^e\| \leq \delta$$

implies that

$$\|x(t; x(0), 0) - x^e\| \leq \varepsilon \text{ for all } t \geq 0.$$

Roughly speaking, an equilibrium state x^e is stable if the response due to any initial state that is sufficiently near to x^e will not move far away from x^e. If the response will, in addition, go back to x^e, then x^e is said to be asymptotically stable.

DEFINITION 4. An equilibrium state x^e is said to be asymptotically stable if it is stable in the sense of Lyapunov and if every motion starting sufficiently near to x^e converges to x^e as $t \to \infty$.

Let us denote

$$s_i = \mathrm{Re}(s_i) + j\mathrm{Im}(s_i),\ i = 1, 2, 3, \ldots, r,\ r \leq n,$$

the distinct eigenvalues of the matrix A and let "Re" and "Im" stand for the real part and the imaginary part of the eigenvalue s_i, respectively.

THEOREM 2. *Every zero state of the dynamical system* (1) *is stable if and only if all eigenvalues of the matrix A have nonpositive* (*negative or zero*) *real parts*, *i.e.*, $\mathrm{Re}(s_i) \leq 0$ *for* $i = 1, 2, 3, \ldots, r$ *and those with zero real parts are simple zeros of the minimal polynomial of the matrix A.*

THEOREM 3. *The zero state of the dynamical system* (1) *is asymptotically stable if and only if all eigenvalues of the matrix A have negative real parts i.e.*,

$$\mathrm{Re}(s_i) \leq 0 \ for \ i = 1, 2, 3, \ldots, r.$$

From the above Theorems directly follows, that stability and asymptotic stability of a dynamical system depend only on the matrix A and are independent of the matrices B, and C.

Suppose that the dynamical system (1) is stable or asymptotically stable, then the dynamical system remains stable or asymptotically stable after arbitrary equivalence transformation. This is natural and intuitively clear because an equivalence transformation changes only the basis of the state space. Therefore, we have the following Corollary.

COROLLARY 5. *Stability and asymptotic stability are both invariant under any equivalence transformation.*

It is well known, that the controllability concept for linear dynamical system (1) is strongly related to its stabilizability by the linear static state feedback of the following form

$$u(t) = Kx(t) + v(t) \tag{5}$$

where $v(t) \in R^m$ is a new control vector and K is $m \times n$-dimensional constant state feedback matrix.

Introducing the linear static state feedback given by equality (5) we directly obtain new linear differential state equation for the feedback linear dynamical system of the following form

$$x'(t) = (A + BK)x(t) + Bv(t) \tag{6}$$

which is characterized by the pair of constant matrices $(A + BK,\ B)$.

An interesting result is the equivalence between controllability of the dynamical systems (1) and (6), explained in the following Corollary.

COROLLARY 6. *Dynamical system* (1) *is controllable if and only if for any arbitrary matrix K the dynamical system* (6) *is controllable.*

From Corollary 6 it follows that under controllability assumption we can arbitrarily form the spectrum of dynamical system (1) by the introduction of suitable defined linear static state feedback (5). Hence, we have the following result.

THEOREM 4. *The pair of matrices* (A, B) *represents the controllable dynamical system* (1) *if and only if for each set* Λ *consisting of* n *complex numbers and symmetric with respect to real axis*, *there exists constant state feedback matrix* K *such that the spectrum of the matrix* $(A + BK)$ *is equal to the set* Λ.

Practically, in the design of the dynamical system, sometimes it is required only to change unstable eigenvalues (i.e., the eigenvalues with nonnegative real parts) into stable eigenvalues (i.e., the eigenvalues with negative real parts). This is called stabilization of the dynamical system (1). Therefore, we have the following formal definition of stabilizability.

DEFINITION 5. Dynamical system (1) is said to be stabilizable if there exists a constant static state feedback matrix K such that the spectrum of the matrix $(A+BK)$ entirely lies in the left-hand side of the complex plane.

Let $\mathrm{Re}(s_j) \geq 0$ for $j = 1, 2, 3, \ldots, q \leq n$, i.e. s_j, are unstable eigenvalues of the dynamical system (1). An immediate relation between controllability and stabilizability of dynamical system (1) gives the following Theorem.

THEOREM 5. *Dynamical system* (1) *is stabilizable if and only if all its unstable modes are controllable i.e.*,

$$rank\,[s_j I - A|B] = n \text{ for } j = 1, 2, 3, \ldots, q.$$

Comparing Theorem 5 and Corollary 3 we see, that controllability of dynamical system (1) always implies its stabilizability, but the converse statement is not always true. Therefore, stabilizability concept is essentially weaker than the controllability property.

3.3. *Output controllability.* Similar to the state controllability of dynamical control system, it is possible to define the so-called output controllability for the output vector $y(t) \in R^p$ of dynamical system. Although these two concepts are quite similar, it should be mentioned that the state controllability is a property of the differential state equation (1), whereas the output controllability is a property both of the state equation (1) and algebraic output equation (2).

DEFINITION 6. Dynamical system (1), (2) is said to be output controllable if for every $y(0)$ and every vector $y^1 \in R^p$, there exist a finite time t_1 and control $u^1(t) \in R^m$, that transfers the output from $y(0)$ to $y^1 = y(t_1)$.

Therefore, output controllability generally means, that we can steer output of dynamical system independently of its state vector.

Theorem 6. *Dynamical system* (1), (2) *is output controllable if and only if*

$$rank\,[CB|CAB|CA^2B|\dots|CA^kB|\dots|CA^{n-1}B] = p.$$

It should be pointed out, that the state controllability is defined only for the linear differential state equation (1), whereas the output controllability is defined for the input-output description i.e., it depends also on the linear algebraic output equation (2). Therefore, these two concepts are not necessarily related.

Let us recall, that if the control system is output controllable, its output can be transferred to any desired vector at certain instant of time. However, a related problem is whether it is possible to steer the output following a given curve over any interval of time. A control system whose output can be steered along the arbitrary given curve over any interval of time is said to be output function controllable or functional reproducible. Conditions for function output controllability are essentially more restrictive than for output controllability.

3.4. *Controllability with Constrained Controls.* In practice admissible controls are required to satisfy additional constraints. Let $U \subset R^m$ be an arbitrary set and let the symbol $M(U)$ denotes the set of admissible controls, i.e., the set of controls $u(t) \in U$ for $t \in [0, \infty]$.

Definition 7. Dynamical system (1) is said to be U-controllable to zero if for any initial state $x(0) \in R^n$ there exist a finite time $t_1 < \infty$ and an admissible control $u(t) \in M(U)$, $t \in [0, t_1]$, such that $x(t_1; x(0), u) = 0$.

Definition 8. Dynamical system (1) is said to be U-controllable from zero if for any final state $x^1 \in R^n$ there exist a finite time $t_1 < \infty$ and an admissible control $u(t) \in M(U)$, $t \in [0, t_1]$, such that $x(t_1; 0, u) = x^1$.

Definition 9. Dynamical system (1) is said to be U-controllable if for any initial state $x(0) \in R^n$, and any final state $x^1 \in R^n$, there exist a finite time $t_1 < \infty$ and an admissible control $u(t) \in M(U)$, $t \in [0, t_1]$, such that $x(t_1; x(0), u) = x^1$.

Generally, for arbitrary given set U it is rather difficult to give easily computable criteria for constrained controllability. However, for certain special cases of the set U it is possible to formulate and prove algebraic constrained controllability conditions.

Theorem 7. *Dynamical system* (*1*) *is U-controllable to zero if and only if all the following conditions are satisfied simultaneously*:

(1) *there exists $w \in U$ such that $Bw = 0$*,

(2) *the convex hull conv* (U) *of the set* U *has nonempty interior in the space* R^m,

(3) *rank* $[B|AB|A^2B|\dots|A^kB|\dots|A^{n-1}B] = n$,

(4) *there is no real eigenvector* $v \in R^n$ *of the matrix* A^T *satisfying* $v^TBw \leq 0$ *for all* $w \in U$,

(5) *no eigenvalue of the matrix* A *has a positive real part.*

For the single input system i.e., $m = 1$, Theorem 7 reduces to the following Corollary.

COROLLARY 7. *Suppose that* $m = 1$ *and* $U = [0, 1]$. *Then dynamical system* (1) *is* U*-controllable to zero if and only if it is controllable without any constraints i.e.,*

$$rank\ [B|AB|A^2B|\dots|A^kB|\dots|A^{n-1}B] = n$$

and matrix A *has no real eigenvalues.*

THEOREM 8. *Suppose the set* U *is a cone with vertex at zero and a nonempty interior in the space* R^m. *Then dynamical system* (1) *is* U*-controllable from zero if and only if*

(1) *rank* $[B|AB|A^2B|\dots|A^kB|\dots|A^{n-1}B] = n$,

(2) *there is no real eigenvector* $v \in R^n$ *of the transposed matrix* A^T *satisfying*

$$v^TBw \leq 0 \text{ for all } w \in U.$$

In the special case for the single input system i.e., $m = 1$, Theorem 8 reduces to the following simple Corollary.

COROLLARY 8. *Suppose that* $m = 1$ *and* $U = [0, 1]$. *Then dynamical system* (1) *is* U*-controllable from zero if and only if it is controllable without any constraints*; *in other words*, *rank* $[B\ A\ B\ A^2B\ \dots\ A^{n-1}B] = n$ *and matrix* A *has no real eigenvalues.*

3.5. *Controllability after the introducing of sampling.* We consider now the case in which the control vector $u(t)$ is piecewise constant i.e., the control $u(t)$ changes value only at a given discrete instant of time. Inputs of this type occur in sampled-data dynamical systems or in dynamical systems in which digital computers are used to generate the control vector $u(t)$. A piecewise-constant control $u(t)$ is often generated by a sampler and a filter, called zero-order hold. In this case we have

$$u(t) = u(k) \text{ for } kT \leq t < (k+1)T,\ k = 0, 1, 2, \dots$$

where T is a positive constant, called the sampling period. The discrete times $0, T, 2T, \dots$ are called sampling instant.

The behavior at sampling instant $0, T, 2T, \ldots$ of the dynamical system (1), (2) with the piecewise-constant inputs are described by the discrete-time linear difference state equations and output equations

$$x(k+1) = Ex(k) + Fu(k) \tag{7}$$

$$y(k) = Cx(k) \tag{8}$$

where constant matrices E and F of appropriate dimensions can be computed using solution of the differential state equation (1) as follows

$$E = e^{AT}, \quad F = \left(\int_0^T e^{sA} ds \right) B$$

In the case when the continuous-time dynamical system (1) is controllable, it is of interest to study whether the dynamical system remains controllable after the introducing of sampling or equivalently, whether discrete-time dynamical system (7) is also controllable. This problem is solved in the next theorem.

THEOREM 9. *Let us assume that the dynamical system (1) is controllable. Then the discrete-time system* (7) *is also controllable if*

$$\mathrm{Im}(s_i - s_j) \neq 2\pi q/T \quad \textit{for} \quad q = \ldots -2, -1, +1, +2, \ldots$$
$$\textit{whenever} \quad \mathrm{Re}(s_i - s_j) = 0.$$

Hence, condition given in Theorem 9 is only a sufficient condition for controllability of discrete-time dynamical system (7). However, in special case when the input $u(t)$ is scalar function we have the following Corollary.

COROLLARY 9. *For the single-input case i.e., for* $m = 1$, *the condition stated in Theorem 9 is necessary as well.*

Finally, for dynamical systems with only real eigenvalues the following two simple Corollaries can be stated.

COROLLARY 10. *If the dynamical system* (1) *is controllable and has only real eigenvalues i.e.,*

$$\mathrm{Im}(s_i) = 0 \quad \textit{for } i = 1, 2, 3, \ldots, r,$$

then the discrete-time system (7) *is always controllable.*

COROLLARY 11. *Dynamical system* (1) *with single input and only real eigenvalues is controllable if and only if the discrete-time dynamical system* (7) *is controllable.*

3.6. *Perturbations of controllable dynamical systems.* In practice the fundamental problem is the question, which bounded perturbations of the parameters of the dynamical system (1) preserve controllability. This problem is explained in the next Theorem.

THEOREM 10. *Suppose that dynamical system* (1) *is controllable. Then there exists* $\varepsilon > 0$ *such that if*

$$\|A - F\| + \|B - G\| < \varepsilon \tag{9}$$

then the dynamical system of the form

$$z'(t) = Fz(t) + Gu(t), \quad z(t) \in R^n$$

is also controllable for any constant matrices F *and* G *of appropriate dimensions, satisfying inequality* (9).

Let us observe that Theorem 10 can be used in the investigations of the topological properties of the set of controllable systems. It should be pointed out, that we can consider dynamical system (1) as a point in the space of parameters $R^{n(n+m)}$.

COROLLARY 12. *For given dimensions* n *and* m *the set of dynamical systems, which are controllable, is open and dense in the space* $R^{n(n+m)}$ *of all dynamical system of the form (1).*

Corollary 12 is of great practical importance. It states that almost all dynamical systems of the form (1) are controllable. Therefore, controllability is the so-called "generic" property of the dynamical system (1). Intuitively this means that almost all dynamical systems are controllable and moreover, that for almost all dynamical systems there exist open neighborhoods containing entirely only controllable dynamical systems. Moreover, Corollary 12 enables us to define the so-called controllability margin. The controllability margin for dynamical system (1) is defined as the distance in the space $R^{n(n+m)}$ between the given dynamical system and the nearest uncontrollable dynamical system. It is obvious that dynamical system (1), which is not controllable, has the controllability margin equal to zero.

4. Minimum energy control. Minimum energy control problem is strongly related to controllability problem. For controllable dynamical system (1) there exists generally many different controls which steer the system from a given initial state $x(0)$ to the final desired state x^1 at time $t_1 > 0$. Therefore, we may look for the control which is an optimal in the sense of the following performance index.

$$J(u) = \int_0^{t_1} \|u(t)\|_Q^2 \, dt$$

where

$$\|u(t)\|_Q^2 = u^T(t)Qu(t)$$

and Q is an $m \times m$-dimensional constant symmetric and positive definite weighting matrix.

The performance index $J(u)$ defines the control energy in the time interval $[0, t_1]$ with the weight determined by the matrix Q. The control u, which minimizes the performance index $J(u)$ is called the minimum energy control. It should be mentioned, that the performance index $J(u)$ is a special case of the general quadratic performance index, and hence the existence of a minimizing control function is assured.

Therefore, the minimum energy control problem can be formulated as follows: for a given arbitrary initial state $x(0)$, arbitrary final state x^1, and finite time $t_1 > 0$, find an optimal control $u(t)$, $t \in [0, t_1]$, which transfers the state $x(0)$ to x^1 at time t_1 and minimizes the performance index $J(u)$.

In order to solve the minimum control problem and to present it in a readable compact form, let us introduce the following notation:

$$W_Q = \int_0^{t_1} e^{tA} B Q^{-1} B^T \left(e^{tA}\right)^T dt,$$

W_Q is constant $n \times n$-dimensional symmetric matrix

$$u^0(t) = Q^{-1} B^T \left(e^{(t_1 - t)A}\right)^T W_{Q^{-1}} [x^1 - e^{t_1 A} x(0)] \tag{10}$$

Exact analytical solution of the minimum energy control problem for dynamical system (1) is given by the Theorem 11, which is proved under following assumptions:
1. Dynamical system is linear,
2. There are no constraints in control,
3. There are no constraints posed on state variable $x(t)$,
4. Dynamical system is controllable,
5. Performance index $J(u)$ does not contain state variable $x(t)$,
6. Performance index is a quadratic with respect to control $u(t)$.

THEOREM 11. *Let $u^1(t)$, $t \in [0, t_1]$ be any control that transfers initial state $x(0)$ to final state x^1 at time t_1, and let $u^0(t)$, $t \in [0, t_1]$ be the control defined by equality* (10). *Then the control $u^0(t)$ transfers the initial state $x(0)$ to a final state x^1 at time t_1 and*

$$J(u^1) \geq J(u^0).$$

Moreover, the minimum value of the performance index corresponding to the optimal control u^0 is given by the following formula

$$J(u^0) = [x^1 - e^{t_1 A} x(0)]^T W_{Q^{-1}} [x^1 - e^{t_1 A} x(0)]$$

5. Controllability of infinite dimensional systems. In the literature there are many different definitions of controllability which depend on the type of dynamical system [1], [5], [8], [9]. A growing interest has been developed over the past few years in problems involving signals and systems that are defined in infinite-dimensional linear spaces. The majority of the results in this area concern linear systems with constant coefficients.

It should be pointed out that for linear systems controllability conditions have pure algebraic forms and are rather easily computable. These conditions require verification of the rank conditions for suitable defined constant controllability matrices.

The most popular examples of infinite-dimensional dynamical systems are distributed parameter systems and dynamical systems with different types of delays in state variables.

5.1. *Mathematical model.* In this section we study the linear infinite dimensional control system with constant coefficients described by abstract differential state equation

$$x'(t) = Ax(t) + Bu(t) \quad \text{for } t \in [0,\ T] \tag{11}$$

with initial condition

$$x(0) \in X \tag{12}$$

where the state $x(0)$ takes values in a real infinite-dimensional separable Hilbert space X and the values of the control $y(t)$ are in the space $U = R^m$.

Let us assume that the linear, generally unbounded, operator A generates a strongly differentiable semigroup $S(t)$ on X for $t \geq 0$ and B is a linear bounded operator from the space R^m into X. Therefore, operator $B = [b_1 b_2 \ldots b_j \ldots \mathrm{b_m}]$ and

$$Bu(t) = \sum_{j=1}^{m} b_j u_j(t)$$

where

$$b_j \in X \quad \text{for } j = 1, 2, 3, \ldots, m,$$
$$u(t) = [u_1(t), u_2(t), \ldots, u_j(t), \ldots, u_m(t)]^T.$$

We would like to emphasize that the assumption that linear operator B is bounded, rules out the application of our theory to boundary control problems, because in this situation B is typically an unbounded operator.

Let $U_c \subset U$ be a closed convex cone with nonempty interior and vertex at zero. The set of admissible controls for the dynamical system (1) is $U_{ad} = L_\infty([0, T], U_c)$.

Then for a given admissible control $u(t)$ there exists a unique so-called mild solution $x(t; x(0), u)$ of the equation (1) with initial condition (2) described by the integral formula

$$x(t;x(0),u) = S(t)x(0) + \int_0^t S(t-s)Bu(s)ds.$$

5.2. *Controllability conditions.* For the linear abstract dynamical system (1), it is possible to define many different concepts of controllability. In the sequel we shall focus our attention on the so-called constrained exact controllability in the time interval $[0,T]$. In order to do that, first of all let us introduce the notion of the attainable set at time $T > 0$ from the zero initial state $x(0) = 0$, denoted by $K_T(U_c)$ and defined as follows

$$K_T(U_c) = \{x \in X : x = x(T,0,u), u(t) \in U_c \text{ for a.e. } t \in [0,T]\},$$

where $x(t,0,u)$ for $t > 0$ is the unique solution of the equation (11) with zero initial condition and control u. Moreover, let us denote

$$K_\infty(U_c) = \bigcup_{t>0} K_t(U_c)$$

Now, using the above concepts of the attainable sets, let us recall the familiar definitions of constrained exact controllability for dynamical system (11).

Definition 10. Dynamical system (11) is said to be U_c-approximately controllable in $[0,T]$ if the attainable set $K_T(U_c)$ is dense in the space X.

Definition 11. Dynamical system (11) is said to be U_c-approximately controllable if the attainable set $K_\infty(U_c)$ is dense in the space X.

Definition 12. Dynamical system (11) is said to be U_c-exactly controllable in $[0,T]$ if $K_T(U_c) = X$.

Definition 13. Dynamical system (11) is said to be U_c-exactly controllable if $K_\infty(U_c) = X$.

Approximate controllability in $[0,T]$ implies approximate controllability and similarly exact controllability in $[0,T]$ implies exact controllability. Moreover, exact controllability always implies approximate controllability. However, conditions for exact controllability are rather very restrictive, therefore in the sequel only approximate controllability will be considered.

Let us observe, that for the finite-dimensional case i.e., when the state space $X = R^n$, we may omit the words "approximate" and "exact" in the above definitions since in this case exact controllability is equivalent to approximate controllability.

In order to obtain computable criteria for approximate controllability we shall concentrate on dynamical systems defined in a separable infinite-dimensional Hilbert space X with a normal, generally unbounded, operator A with compact resolvent. Hence, the operator A has only pure discrete point spectrum consisting of an infinite sequence $\{s_i\}$, $i = 1, 2, 3, \ldots$, of distinct isolated eigenvalues of A, each with finite multiplicity r_i. Moreover, in the space X there is a corresponding complete orthonormal set $\{x_{ij}\}$, $i = 1, 2, 3, \ldots, j = 1, 2, 3, \ldots, r_i$, of eigenvectors of the operator A. Therefore, the semigroup $S(t)$ is given by

$$S(t)x = \sum_{i=1}^{\infty} e^{s_i t} \sum_{j=1}^{r_i} \langle x, x_{ij} \rangle_X \, x_{ij}, \quad \text{for } t \geq 0 \text{ and } x \in X,$$

where symbol $\langle \cdot, \cdot \rangle_X$ denotes scalar product in Hilbert space X.

The class of operators satisfying the above assumptions arises in classical control problems for linear distributed parameters systems.

In order to formulate computable, constrained approximate controllability conditions let us denote B_i, $i = 1, 2, 3, \ldots$, $(r_i \times m)$-dimensional constant matrices,

$$B_i = \begin{bmatrix} b_{1i1} & b_{2i1} & \ldots & b_{ki1} & \ldots & b_{mi1} \\ b_{1i2} & b_{2i2} & \ldots & b_{ki2} & \ldots & b_{mi2} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{1ij} & b_{2ij} & \ldots & b_{kij} & \ldots & b_{mij} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{1ir_i} & b_{2ir_i} & \ldots & b_{kir_i} & \ldots & b_{mir_i} \end{bmatrix} \tag{13}$$

where $b_{kij} = \langle b_k, x_{ij} \rangle_X$ for $i = 1, 2, 3, \ldots, j = 1, 2, 3, \ldots, r_i, k = 1, 2, 3, \ldots, m$.

For the case when all the eigenvalues s_i are simple i.e., $r_i = 1$, for $i = 1, 2, 3, \ldots$, B_i are m-dimensional row vectors of the following form

$$B_i = [\langle b_1, x_i \rangle_X, \langle b_2, x_i \rangle_X, \ldots, \langle b_k, x_i \rangle_X, \ldots, \langle b_m, x_i \rangle_X] \quad \text{for } i = 1, 2, 3, \ldots \tag{14}$$

Theorem 12. *Let X be a separable Hilbert space and assume that the operator A is normal with compact resolvent. Let $U = R^m$ and U_c be a cone in R^m with vertex at the origin and such that int(conv U_c) $\neq \emptyset$. Then the linear dynamical system* (11), (12) *is approximately U_c-controllable in finite time if and only if the following conditions hold*

$$B_i conv(U_c) = R^r \quad \text{for all } i = 1, 2, 3, \ldots$$

whenever s_i is a real eigenvalue

$$rank\, B_i = r_i \quad \text{for } i = 1, 2, 3, \ldots$$

whenever s_i is a complex eigenvalue.

Corollary 13. *Let $U = U_c = R^m$. Then the linear dynamical system* (1) *is approximately R^m-controllable in finite time if and only if the following conditions hold*

$$rank\, B_i = r_i \quad \text{for } i = 1, 2, 3, \ldots$$

Corollary 14. *Let* $U = U_c = R^m$ *and* $r_i = 1$ *for* $i = 1, 2, 3, \ldots$ *Then the linear dynamical system* (11) *is approximately* R^m*-controllable in finite time if and only if the following conditions hold*

$$\sum_{j=1}^{m} \langle b_j, x_i \rangle^2_{L^2(D)} \neq 0 \quad \textit{for } i = 1, 2, 3, \ldots$$

or equivalently

$$B_i \neq 0 \quad \textit{for } i = 1, 2, 3, \ldots$$

It should be pointed out that the multiplicity's r_i of the eigenvalues s_i, $i = 1, 2, 3, \ldots$, are finite for every index i, however, we do not always have $\sup r_i < \infty$. The number $r = \sup r_i < \infty$ has an important meaning in the investigation of approximate controllability.

Corollary 15. *If* $r = \sup r_i = \infty$ *then the dynamical system* (1) *is not* R^m*-approximately controllable.*

The quite general controllability conditions given in Theorem 1 and Corollaries 1 and 2, can be used to formulate controllability criteria for distributed parameter dynamical systems described by the linear partial differential state equations.

6. Controllability of distributed parameters systems.

6.1. *Mathematical model.* The motivation for studying distributed parameters systems has been well justified in several papers and monographs. Most of the major results concerning infinite dimensional dynamical systems are developed for linear case. During the last two decades controllability of distributed parameter systems have been considered in many papers and books. The main purpose of this section is to present a compact review over the existing controllability results mainly for linear distributed parameter dynamical systems.

Many time-independent linear distributed parameter systems can be represented in the framework of the abstract differential equation (11), with operator A corresponding to the “spatial” differential operator appearing in equation (11). Therefore, the general results concerning approximate controllability of dynamical systems defined in infinite-dimensional linear spaces can be used to analyze approximate controllability of distributed parameter systems described by partial differential equations of parabolic type in several space dimensions.

Let us consider the distributed parameter dynamical system defined in a bounded domain $D \subset R^m$, with sufficiently smooth boundary S, described by the time-invariant partial differential equation of parabolic type of the following form:

$$\frac{\partial w(z,t)}{\partial t} = \sum_{k=1}^{n} \frac{\partial^2 w(z,t)}{\partial z_k^2} + q(z)w(z,t) + \sum_{j=1}^{m} b_j u_j(t) \quad \text{for } z \in D,\ t > 0 \tag{15}$$

with boundary condition

$$\frac{\partial w(z,t)}{\partial \nu} + p(z)w(z,t) = 0 \quad \text{for } z \in S,\ t > 0 \tag{16}$$

where ν is a vector normal at S exterior to D, and with the initial condition

$$w(z,0) = w_0(z),\ z \in D,\ w_0(z) \in L^2(D). \tag{17}$$

It is assumed that function $q(z)$ is continuous in the set $D \cup S$, and function $p(z)$ is continuous in the set S.

The dynamical system can be connected with a linear unbounded differential operator $A : L^2(D) \supset D(A) \to L^2(D)$ defined as follows

$$Aw(z) = \sum_{k=1}^{n} \frac{\partial^2 w(z)}{\partial z_k^2} + q(z)w(z) \tag{18}$$

where $w(z) \in D(A)$, $D(A) = \{w(z) \in L^2(D) : Aw(z) \in L^2(D)\}$,

$$\frac{\partial w(z,t)}{\partial \nu} + p(z)w(z,t) = 0,\ z \in S. \tag{19}$$

The domain $D(A)$ of the operator A is dense in the separable Hilbert space $L^2(D)$. Moreover, since the set D is bounded then operator A satisfies all the assumptions stated in subsection 5.1, i.e. has only pure discrete point spectrum consisting of an infinite sequence $\{s_i\}$, $i = 1, 2, 3, \ldots$, of distinct isolated eigenvalues of A, each with finite multiplicity r_i. Moreover, in Hilbert space $L^2(D)$ there is a corresponding complete orthonormal set $\{x_{ij}(z)\}$, $i = 1, 2, 3, \ldots$, $j = 1, 2, 3, \ldots, r_i$, $z \in D$, of eigenvectors of the operator A.

6.2. *Controllability conditions.* Using the general controllability results given in subsection 5.2 it is possible to formulate necessary and sufficient conditions for approximate controllability of the distributed parameter system (15).

THEOREM 13. *Let us assume that the operator A satisfies all the assumptions stated above, and let U_c be a cone in R^m with vertex at the origin such that $int(conv\, U_c) \neq \emptyset$. Then the linear distributed parameter dynamical system* (15) *with boundary conditions* (16) *is approximately U_c-controllable in finite time if and only if the following conditions hold*

$$B_i conv\,(U_c) = R^r \quad \textit{for all } i = 1, 2, 3, \ldots,\ \textit{whenever } s_i \textit{ is a real eigenvalue},$$

$$rank\ B_i = r_i \quad \textit{for } i = 1, 2, 3, \ldots,\ \textit{whenever } s_i \textit{ is a complex eigenvalue},$$

where B_i, $i = 1, 2, 3, \ldots$, *are* $(r_i \times m)$*-dimensional constant matrices given by the equalities* (13) *with inner product*

$$\langle b_j(z), x_{ik}(z)\rangle_{L^2(D)} = \int_D b_j(z)x_{ik}(z)dz \tag{20}$$

Corollary 16. *Let us assume that the operator* A *satisfies all the assumptions stated above and* $U = U_c = R^m$*. Then the distributed parameter dynamical system* (15) *with boundary conditions* (16) *is* R^m*-approximately controllable if and only if*

$$rank\ B_i = r_i \quad for\ i = 1, 2, 3, \ldots,$$

where B_i, $i = 1, 2, 3, \ldots$, *are* $(r_i \times m)$*-dimensional constant matrices given by the equalities* (3) *with inner product given by equality* (20).

Corollary 17. *Let* $U = U_c = R^m$ *and* $r_i = 1$, *for* $i = 1, 2, 3, \ldots$. *Then the linear dynamical system* (15) *with boundary conditions* (16) *is approximately* R^m*-controllable in finite time if and only if the following conditions hold*

$$\sum_{j=1}^{m} \langle b_j, x_i\rangle^2_{L^2(D)} \neq 0 \quad for\ i = 1, 2, 3, \ldots$$

or equivalently $B_i \neq 0$ *for* $i = 1, 2, 3, \ldots$ *where* B_i, $i = 1, 2, 3, \ldots$ *are* m*-dimensional constant vectors given by the equalities* (14) *with inner product*

$$\langle b_j(z), x_{ik}(z)\rangle_{L^2(D)} = \int_D b_j(z)x_{ik}(z)dz.$$

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Sterowalność układów dynamicznych

Streszczenie. Sterowalność, podobnie jak obserwowalność oraz stabilność, należy do podstawowych pojęć matematycznej teorii układów dynamicznych. Ogólnie mówiąc, sterowalność oznacza, że w rozpatrywanym układzie dynamicznym możliwe jest osiągnięcie zadanego stanu końcowego przy użyciu odpowiednio dobranego sterowania dopuszczalnego, należącego do zadanego zbioru sterowań dopuszczalnych. Zatem sterowalność zależy w istotny sposób zarówno od modelu matematycznego układu dynamicznego reprezentowanego równaniem stanu, jak i od postaci zbioru sterowań dopuszczalnych.

Pojęcie sterowalności układu dynamicznego jest wykorzystywane między innymi do analizowania i tworzenia tak zwanych form kanonicznych układów dynamicznych oraz przy formułowaniu twierdzeń z zakresu sterowania optymalnego. Odgrywa ono również istotną rolę w teorii gier oraz w analizie jakościowej układów dynamicznych. Do analizowania problematyki sterowalności układów dynamicznych wykorzystuje się metody zaczerpnięte z różnych, często odległych od siebie dziedzin matematyki, między innymi takich jak: algebra, analiza funkcjonalna, równania różniczkowe, teoria optymalizacji.

W artykule, wykorzystując metody algebraiczne, sformułowano kryteria badania sterowalności dla liniowych, skończenie-wymiarowych, ciągłych układów dynamicznych o stałych współczynnikach zarówno dla przypadku braku ograniczeń, jak i dla stożkowo ograniczonych wartości sterowań. Rozpatrzono związki zachodzące pomiędzy sterowalnością a stabilizowalnością układu dynamicznego. Zakładając sterowalność układu, podano także analityczną postać rozwiązania zagadnienia sterowania z minimalną energią.

W drugiej części artykułu, w oparciu o spektralną teorię liniowych operatorów różniczkowych oraz twierdzenia z zakresu analizy funkcjonalnej, sformułowano warunki konieczne i wystarczające aproksymacyjnej sterowalności dla liniowych układów dynamicznych o parametrach rozłożonych.
Słowa kluczowe: liniowe układy dynamiczne, sterowalność, stabilność, układy dynamiczne o parametrach rozłożonych.

(*wpłynęło 7 grudnia 2007 r.*)