Math 211 **Final Examination questions** **Name:**___________________

Answer 10 out of 11 questions. Ten points for each question.

1. For each of the following matrices, determine whether it is diagonalizable (with explanation).

   (a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$,

   (b) $B = \begin{bmatrix} 0 & 4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

2. Suppose $A = \begin{bmatrix} 2 & 3 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$, and $\det(A - \lambda I) = (1+\lambda)^2(5-\lambda)$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

3. Let $T : \mathbf{P}_1(t) \to \mathbf{P}_1(t)$ defined by $T(a_0 + a_1 t) = (4a_0 - 2a_1) + (a_0 + a_1)t$.

(a) Find the matrix for T relative to the standard basis $\mathcal{C} = \{1, t\}$.

(b) Find eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{u}_1, \mathbf{u}_2 \in \mathbf{P}_1(t)$ such that $T(\mathbf{u}_i) = \lambda_i \mathbf{u}_i$ for $i = 1, 2$.

4. Let $\mathbf{p}(t) = t^2$, $\mathbf{q}(t) = 3t - 4$ and $\mathbf{r}(t) = 4t + 3$ and $T : \mathbf{R}^3 \to \mathbf{P}_2(t)$ such that

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = (2a + 5b - 3c)\mathbf{p}(t) + (3b - 3c)\mathbf{q}(t) + (-4a + b - 5c)\mathbf{r}(t)$$

(a) Determine with explanation the matrix of T relative to the bases $\{e_1, e_2, e_3\}$ and $\{1, t, t^2\}$.

(b) Determine with explanation the matrix of T relative to $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right\}$ and $\{\mathbf{p}(t), \mathbf{q}(t), \mathbf{r}(t)\}$.

5. Let $W = \left\{ \begin{bmatrix} b-a \\ -3b \\ a+b \end{bmatrix} : a, b \in \mathbf{R} \right\}$. (a) Is $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}$ in the orthogonal complement of W?

(b) Write $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{y}_1 + \mathbf{y}_2$ such that $\mathbf{y}_1 \in W$ and $\mathbf{y}_2 \in W^{\perp}$.

6. Find an **orthonormal** basis for $(\mathrm{Col}(A))^{\perp}$, where $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & 0 & 1 \\ 10 & -6 & 4 \end{bmatrix}$.

7. Let $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$. Find the least-squares solution of $A\mathbf{x} = \mathbf{b}$, and compute the associated least-squares error.

8. Let A be an $m \times n$ matrix.

(a.1) If $\mathbf{x} \in \text{Nul}A$, show that $\mathbf{x} \in \text{Nul}(A^T A)$; (a.2) If $\mathbf{x} \in \text{Nul}(A^T A)$, show that $\mathbf{x} \in \text{Nul}A$.

(b) Show that $A^T A$ and A has the same rank. [Hint: $\text{rank}\, A + \dim \text{Nul}\, A = n$.]

9. Let $\mathbf{u}, \mathbf{v} \in \mathbf{R}^n$. (a) Show that $\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2$.

(b) If $\{\mathbf{u}, \mathbf{v}\}$ is **orthonormal**, show that $\{\frac{1}{\sqrt{2}}(\mathbf{u}+\mathbf{v}), \frac{1}{\sqrt{2}}(\mathbf{u}-\mathbf{v})\}$ is also **orthonormal**.

10. (a) Suppose A and B are 2×2 matrices, and both of them have eigenvalues 1 and -1. Show that there are invertible matrices P, Q, R such that $P^{-1}AP = Q^{-1}BQ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R^{-1}AR = B$.

(b) Give an example of a pair of 2×2 matrices C and D such that $\det(C - \lambda I) = (1-\lambda)^2 = \det(D - \lambda I)$, but C and D are not similar.

11. Suppose W is a subspace of $\mathbf{R}^n$ with an **orthonormal** basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ and $W^\perp$ has an **orthonormal** basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_q\}$.

(a) Show that $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ is an **orthonormal** set.

(b) Show that $\mathcal{B}$ is a basis for $\mathbf{R}^n$, and deduce that $p + q = n$.

Question	1	2	3	4	5	6	7	8	9	10	11	Total
Score												