Sample Solution.

Math 211 **Final Examination** **Name:**________________

Answer 10 out of 11 questions. Ten points for each question.

1. For each of the following matrices, determine whether it is diagonalizable (with explanation).

(a) $A = \begin{bmatrix} 0 & 4 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

No.

Eigenvalues λ: 0, 1.

For $\lambda=0$, $A - 0I = \begin{bmatrix} 0 & 4 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has rank 2. $\therefore$ 1 lin indep eigenvector

For $\lambda=1$, $A - \lambda I = \begin{bmatrix} -1 & 4 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has rank 2 $\therefore$ 1 lin indep eigenvector

$\therefore$ Not enough lin. indep. eigenvectors. (We need 3.)

(b) $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$,

Yes.

Eigenvalues $0 = \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix}$

$= (1-\lambda)\left[(1-\lambda)(1-\lambda) - 1\right] = (1-\lambda)\lambda(\lambda-2)$.

$\therefore$ 3 distinct eigenvalues and there will be 3 linearly independent eigenvectors.

2. Suppose $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}$, and $\det(A - \lambda I) = \lambda^2(6-\lambda)$. Find an invertible matrix P such that $P^{-1}AP$ equal to the diagonal matrix with diagonal entries 0, 0, 6.

Eigenvalues are 0, 6.

For $\lambda=0$, $A - 0I = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Null space = Span $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$

For $\lambda=6$, $A - 6I = \begin{bmatrix} -5 & 1 & 1 \\ 3 & -3 & 3 \\ 2 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 \\ 1 & -1 & 1 \\ -5 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & -3 \\ 0 & 6 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 \\ 0 & -2 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

Null space = Span $\left\{ \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix} \right\}$

$\therefore$ $P = \begin{bmatrix} 1 & 1 & 7 \\ -1 & 0 & -3 \\ 0 & -1 & 2 \end{bmatrix}$ will satisfy

$P^{-1}AP = \begin{bmatrix} 0 & & \\ & 0 & \\ & & 6 \end{bmatrix}$.

3. Let $T : \mathbf{P}_1(t) \to \mathbf{P}_1(t)$ defined by $T(a_0 + a_1 t) = (3a_0 + 4a_1) + (a_0 + 3a_1)t$.

(a) Find the matrix for T relative to the standard basis $\mathcal{C} = \{1, t\}$.

$T(1) = 3 + t, \quad T(t) = 4 + 3t$

$[T(1)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad [T(t)]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$$

(b) Find eigenvalues λ_1, λ_2 and eigenvectors $\mathbf{u} = u_0 + u_1 t, \mathbf{v} = v_0 + v_1 t \in \mathbf{P}_1(t)$ such that $T(\mathbf{u}) = \lambda_1 \mathbf{u}$ and $T(\mathbf{v}) = \lambda_2 \mathbf{v}$

$0 = \det(A - \lambda I) = (3-\lambda)^2 - 4 = \lambda^2 - 6\lambda + 9 - 4 = (\lambda^2 - 6\lambda + 5) = (\lambda - 1)(\lambda - 5).$

For $\lambda = 1$, $\quad A - \lambda I = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$ has null vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ $\quad P^{-1} A P = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

For $\lambda = 5$, $\quad A - 5I = \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix}$ has null vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. $\quad$ with $P = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

Let $u = 2 - t$, $v = 2 + t$. Then $T(u) = 2 - t = 1 \cdot u$, $T(v) = 10 + 5t = 5 \cdot v$.

4. Let $\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ be bases for $\mathbf{R}^3$.

(a) Find the matrix $\mathcal{P}_{\mathcal{C} \leftarrow \mathcal{B}}$.

$$x = \begin{bmatrix} -2 & 3 & 4 \\ 2 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix} [x]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} [x]_{\mathcal{C}}, \quad \therefore P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 3 & 4 \\ 2 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 4 \\ 2 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 3 & 5 \\ 2 & -2 & -4 \\ 0 & 2 & 3 \end{bmatrix}$$

(b) Find $[\mathbf{z}]_{\mathcal{C}}$ if $[\mathbf{z}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$$[z]_{\mathcal{C}} = \begin{bmatrix} -4 & 3 & 5 \\ 2 & -2 & -4 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 5 \end{bmatrix}$$

5. Let $W = \text{Span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$.

(a) Find an orthonormal basis for W and find an orthonormal basis for $W^{\perp}$.

Solving $\begin{bmatrix}1&2&1\\1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$, $W^{\perp} = \text{Span}\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$

o.n. basis is $\left\{\begin{bmatrix}1\\-1\\1\end{bmatrix}/\sqrt{3}\right\}$

Apply Gram-Schmidt to $\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}\right\}$.

$v_1 = \begin{bmatrix}1\\2\\1\end{bmatrix}$, $v_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{3}{6}\begin{bmatrix}1\\2\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\\-1\end{bmatrix}$

So $W = \text{Span}\left\{\begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right\}$

and o.n. basis is $\left\{\frac{\begin{bmatrix}1\\2\\1\end{bmatrix}}{\sqrt{6}}, \frac{\begin{bmatrix}1\\0\\-1\end{bmatrix}}{\sqrt{2}}\right\}$.

(b) Write $\mathbf{y} = \begin{bmatrix}3\\1\\1\end{bmatrix} = \hat{\mathbf{y}} + \mathbf{z}$ such that $\hat{\mathbf{y}} \in W$ and $\mathbf{z} \in W^{\perp}$.

$$\hat{y} = \frac{\begin{bmatrix}1\\2\\1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\1\end{bmatrix}}{\sqrt{6}\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix} + \frac{\begin{bmatrix}1\\0\\-1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\1\end{bmatrix}}{\sqrt{2}\sqrt{2}}\begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} + \begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}2\\2\\0\end{bmatrix}$$

$$z = \frac{\begin{bmatrix}1\\-1\\1\end{bmatrix}\cdot\begin{bmatrix}3\\1\\1\end{bmatrix}}{\sqrt{3}\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix}$$

6. Let $A = \begin{bmatrix}1&2&2\\1&0&2\\1&2&0\\1&0&0\end{bmatrix}$.

(a) Apply the Gram-Schmidt process to the columns of A to get an orthonormal set of vectors in $\mathbf{R}^4$.

(b) Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

(a) $V_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $V_2 = \begin{bmatrix}2\\0\\2\\0\end{bmatrix} - \frac{4}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}$, $V_3 = \begin{bmatrix}2\\2\\0\\0\end{bmatrix} - \begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{0}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}$

$\therefore$ orthonormal set is $\left\{\frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}\right\}$

~~A=~~ $Q = \frac{1}{2}\begin{bmatrix}1&1&-1\\1&-1&-1\\1&1&1\\1&-1&1\end{bmatrix}$

$R = Q^TA = \frac{1}{2}\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\-1&-1&1&1\end{bmatrix}\begin{bmatrix}1&2&2\\1&0&2\\1&2&0\\1&0&0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}4&4&4\\0&4&0\\0&0&-4\end{bmatrix}$

$R = \begin{bmatrix}2&2&2\\0&2&0\\0&0&-2\end{bmatrix}$

7. Suppose A is an 8×6 matrix of rank 5.

(a) What are the dimensions of the column space, row space, and null space of A?

(b) What are the dimensions of the column space, row space, and null space of A^T

(c) Does $\mathrm{Col}(A^T)$ equals $\mathbf{R}^6$? (Give reasons to support your answer.)

(a) dim(col space) = dim(row space) = rank = 5

dim(Null space) = 6 − 5 = 1

(b) dim(Col(A^T)) = dim(Row(A^T)) = rank = 5

dim(Null(A^T)) = 8 − 5 = 3

(c) Col(A^T) has dim 5, So it cannot be $\mathbb{R}^6$.

8. (a) Suppose A, B, C are $n \times n$ matrices such that $ABC = I_n$. Show that $BCA = I_n$.

$ABC = I_n$ Then $A^{-1} = BC$. and $(BC)A = I_n$

(b) Give an example of 2×2 matrices X, Y, Z such that $XYZ = I_2$ but $XZY \neq I_2$.

Let $X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Z = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Then $XYZ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

But $XZY = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & +1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

9. Show that $\left\{\begin{bmatrix} a \\ b \end{bmatrix} : a, b \in \mathbf{R}, ab \geq 0\right\}$ is not a subspace of $\mathbf{R}^2$.

Note that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ belongs to the set

But $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ does not belong to the set.

So the set is not a subspace.

10. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 & 0 \\ 5 & 1 & 0 \\ 7 & 6 & 2 \end{bmatrix}$.

(a) Show that there is an invertible matrix S such that $S^{-1}AS = D$ where D is the diagonal matrix with diagonal entries $1, 2, 3$.

A has eigenvalues 1, 2, 3. So there are 3 linearly independent eigenvectors such that $AX_1 = X_1$, $AX_2 = 2X_2$, $AX_3 = 3X_3$.

Let $S = [X_1 X_2 X_3]$. $S^{-1}AS = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$

(b) Show that there is an invertible matrix T such that $T^{-1}AT = B$.

Note that ~~there~~ B has eigenvalues 1, 2, 3.

So there is P s.t. $P^{-1}BP = \begin{bmatrix} 1 & & \\ & 2 & \\ & & 3 \end{bmatrix}$.

Now $S^{-1}AS = P^{-1}BP$

So $(PS^{-1})A(SP^{-1}) = B$.

Let $T = SP^{-1}$

$T^{-1} = PS^{-1}$.

We have $T^{-1}AT = B$

11. Let A, B, X, Y, Z be $n \times n$ matrices such that A is invertible. Consider the matrix equation

$$\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Find formulas for X, Y, and Z in terms of A and B.

L.S. equals: $\begin{bmatrix} AX & AY & AZ+B \\ 0 & 0 & I \end{bmatrix}$,

$\therefore$ $AX = I$, $\quad X = A^{-1}$ //

$AY = 0$ $\quad Y = A^{-1}0 = 0$ //

$AZ+B = 0$ $\quad Z = A^{-1}(-B) = -A^{-1}B$ //

Question	1	2	3	4	5	6	7	8	9	10	11	Total
Score												