Math 211 Homework 2 **Name: Sample solution**

1 Solving the vector equation

$$x_1 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we consider the augmented matrix

$$\begin{bmatrix} 3 & 7 & -2 & 0 \\ -2 & 3 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 7/3 & -2/3 & 0 \\ 0 & 1 & -1/23 & 0 \end{bmatrix}$$

so that $(x_1, x_2, x_3) = t(13/23, 1/23, 1)$ with $t \in \mathbf{R}$.

2 Writing the system as a vector equation:

$$3x_1 - 2x_2 + 4x_3 = 3,$$
$$-2x_1 - 7x_2 + 5x_3 = 1,$$
$$5x_1 + 4x_2 - 3x_3 = 2,$$

we have

$$x_1 \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -7 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}.$$

We apply row reduction to the augmented matrix to get: $(x_1, x_2, x_3) = (49/73, -18/73, 9/73)$.

3 To find the values of h such that the following vector equation has a solution,

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} h \\ -3 \\ -5 \end{bmatrix}.$$

we consider the augmented matrix and apply row operations:

$$\begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ -2 & 7 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & h \\ -2 & 7 & -5 \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & h \\ 0 & 3 & -5+2h \\ 0 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & h \\ 0 & 1 & -3 \\ 0 & 0 & 2h+4 \end{bmatrix}$$

From this last matrix, we can see that it will be consistent as long as $4 + 2h = 0$, i.e., $h = -2$.

4 Given the following matrix A and a vector $\mathbf{b}$.

$$A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}$$

(a) To determine if $\mathbf{b}$ is in the set W that consists of all linear combinations of the columns of A, we solve the vector equation

$$\begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 10 \\ 3 \\ 7 \end{bmatrix}.$$

We form an augmented matrix and reduce it to row echelon form

$$\begin{bmatrix} 2 & 0 & 6 & 10 \\ -1 & 8 & 5 & 3 \\ 1 & -2 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 7 \\ -1 & 8 & 5 & 3 \\ 2 & 0 & 6 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 6 & 6 & 10 \\ 0 & 4 & 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 1 & 1 & -1 \\ 0 & 3 & 3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 7 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 8 \end{bmatrix}.$$

Because the last row has all zeros except for the rightmost entry, we can see that the system has no solution. Therefore, $\mathbf{b}$ is not a linear combination of columns of A and so does not belong to W.

(b) Let $\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3}$ be the columns of A. Then $\mathbf{a_2} = 0\mathbf{a_1} + 1\mathbf{a_2} + 0\mathbf{a_3}$. So, if $(x_1, x_2, x_3) = (0, 1, 0)$, then $\mathbf{a_2} = x_1\mathbf{a_1} + x_2\mathbf{a_2} + x_3\mathbf{a_3} = \mathbf{a_2}$ is in W.

5 By computing a matrix product, we find that

$$\begin{bmatrix} 1 & 3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 1 \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 1 \cdot \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+3-4 \\ 3+3+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}.$$

6 Given a matrix A and a vector $\mathbf{b}$, to describe the set of all $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has a solution, create an augmented matrix and apply row reductions

$$\begin{bmatrix} 1 & -2 & -1 & b_1 \\ -2 & 2 & 0 & b_2 \\ 4 & -1 & 3 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & -2 & -2 & b_2 + 2b_1 \\ 0 & 7 & 7 & b_3 - 4b_1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 1 & 1 & -\frac{1}{2}(b_2 + 2b_1) \\ 0 & 1 & 1 & \frac{1}{7}(b_3 - 4b_1) \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 1 & 1 & -\frac{1}{2}(b_2 + 2b_1) \\ 0 & 0 & 0 & \frac{1}{7}(b_3 - 4b_1) + \frac{1}{2}(b_2 + 2b_1) \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & b_1 \\ 0 & 1 & 1 & -\frac{1}{2}(b_2 + 2b_1) \\ 0 & 0 & 0 & 6b_1 + 7b_2 + 2b_3 \end{bmatrix}.$$

So, $A\mathbf{x} = \mathbf{b}$ does not have a solution for all possible $\mathbf{b}$. It only has a solution for $\mathbf{b}$ in the set

$$\left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} : 6b_1 + 7b_2 + 2b_3 = 0 \right\} = \left\{ \begin{bmatrix} (-7s - 2t)/6 \\ s \\ t \end{bmatrix} : s, t \in \mathbf{R} \right\} = \left\{ s \begin{bmatrix} -7/6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1/3 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbf{R} \right\}.$$

7 Given three vectors

$$\mathbf{v_1} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 0 \\ -3 \\ 9 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 4 \\ -2 \\ -6 \end{bmatrix},$$

to determine if these vectors generate, or span, $\mathbb{R}^3$, let $\mathbf{b} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$, consider the augmented matrix and apply row reduction:

$$\begin{bmatrix} 0 & 0 & 4 & r \\ 0 & -3 & -2 & s \\ -3 & 9 & -6 & t \end{bmatrix} \to \begin{bmatrix} -3 & 9 & -6 & t \\ 0 & -3 & -2 & s \\ 0 & 0 & 4 & r \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & \frac{-t}{3} \\ 0 & 1 & \frac{2}{3} & \frac{-s}{3} \\ 0 & 0 & 1 & \frac{r}{4} \end{bmatrix}.$$

We see that this system is consistent, which means that an arbitrary vector in $\mathbb{R}^3$ is a linear combination of v_1, v_2, and v_3. Thus, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

8 Given that

$$\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad 2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = 0$$

to find x_1 and x_2 that satisfy

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix},$$

substitute $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ for the columns of the matrices. We get

$$[\mathbf{u} \quad \mathbf{v}] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{w}.$$

By computing the product of the left side, and doing some algebraic rearranging, we find that

$$x_1\mathbf{u} + x_2\mathbf{v} - \mathbf{w} = 0.$$

By looking at the equation that we were given at the beginning of the problem, we see that the appropriate (x_1, x_2) pair is $(2, -3)$.

9 We can write the solution in vector notation as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_4 \\ 3-2x_4 \\ 2+5x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ -2 \\ 5 \\ 1 \end{bmatrix} = \mathbf{p} + x_4\mathbf{v}, \quad \text{where} \quad \mathbf{p} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 5 \\ -2 \\ 5 \\ 1 \end{bmatrix}.$$

This is the line parallel to the vector $\mathbf{v}$ through $\mathbf{p}$.

10 Both $x_1 - 2x_2 + 3x_3 = 0$ and $x_1 - 2x_2 + 3x_3 = 4$ are planes in $\mathbb{R}^3$. The planes are parallel to each other, but $x_1 - 2x_2 + 3x_3 = 0$ goes through the origin whereas $x_1 - 2x_2 + 3x_3 = 4$ is shifted. To determine how much it is shifted. We write the solution in parametric vector form.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4+2x_2-3x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \mathbf{p} + x_2\mathbf{v_1} + x_3\mathbf{v_2}$$

where

$$\mathbf{p} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v_1} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}.$$

By Theorem 6, we know that $\mathbf{p}$ is the vector that shifts the solution of the homogeneous system $(x_2\mathbf{v_1} + x_3\mathbf{v_2})$. Therefore, $x_1 - 2x_2 + 3x_3 = 0$ and $x_1 - 2x_2 + 3x_3 = 4$ are parallel planes, one which goes through the origin, and one which goes through the point $(4, 0, 0)$.

11 (Extra credit problem) Show that an augmented matrix A cannot have two different reduced row echelon form F_1 and F_2.

A complete proof is in Appendix 1 of the textbook, and requires more theory developed in future chapters. Any attempt will earn some partial credits.