Math 211 Homework 3 Sample solution

1. We are given three vectors

$$\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

If we put these vectors into a matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ 2 & 0 & 3 \\ 3 & -8 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & -8 & 1 \\ 2 & 0 & 3 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 3 & -8 & 1 \\ 0 & -16 & -7 \\ 0 & 0 & -1 \end{bmatrix}.$$

From this form, we can clearly see that the system only has one solution, the trivial one (note: this is the coefficient matrix, not the augmented matrix). Thus we see that the vectors are linearly independent.

2. We are given three vectors

$$\mathbf{v_1} = \begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}.$$

If we put these vectors into a matrix

$$\begin{bmatrix} 3 & -6 & 0 \\ -6 & 4 & h \\ 1 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ -6 & 4 & h \\ 1 & -3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & -8 & h+18 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & -8 & h+18 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & h+18 \end{bmatrix}.$$

So, the corresponding homogeneous system has non-trivial solution precisely when $h = -18$, i.e., the set of vectors are linearly dependent precisely when $h = -18$.

3. If the columns of a 3×3 matrix A are linearly independent, then there is no free variables in the augmented matrix in solving the homogeneous system $A\mathbf{x} = \mathbf{0}$. Thus, every column is a pivoting column, and has a pivoting one. In the reduced echelon form, each column with a pivoting one can only have one nonzero entry. So, reduced echelon form of A must be

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. We are given a matrix and two vectors

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 3 \\ 6 \\ -9 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

For the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\mathbf{x}) = A\mathbf{x}$, we have

$$T(\mathbf{u}) = A\mathbf{u} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} - 9 \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = - \begin{bmatrix} 12 \\ 21 \\ 54 \end{bmatrix}$$

and

$$T(\mathbf{v}) = A\mathbf{u} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + c \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 + 2b + 3c \\ 4b + 5c \\ 6c \end{bmatrix}.$$

5. We are given a matrix and a vector

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & -3 \\ 2 & -5 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -6 \\ -4 \\ -5 \end{bmatrix}.$$

To find a vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$, we can create an augmented matrix and reduce it using row operations

$$\begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & -3 & -4 \\ 2 & -5 & 6 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -4 \\ 2 & -5 & 6 & -5 \\ 0 & 1 & -3 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & -1 & 0 & 7 \\ 0 & 1 & -3 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & -6 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Thus the vector $\mathbf{x}$ that satisfies these conditions is the vector

$$\mathbf{x} = \begin{bmatrix} -17 \\ -7 \\ -1 \end{bmatrix}$$

We can also see that this solution is unique because the augmented matrix has no free variables.

6. Let

$$A = \begin{bmatrix} 3 & 2 & 10 & -6 \\ 1 & 0 & 2 & -4 \\ 0 & 1 & 2 & 3 \\ 1 & 4 & 10 & 8 \end{bmatrix}.$$

To find all vectors that are mapped into the zero vector by the transformation $\mathbf{x} \to A\mathbf{x}$, we an augmented matrix and reduce it by row operations.

$$\begin{bmatrix} 3 & 2 & 10 & -6 & 0 \\ 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 1 & 4 & 10 & 8 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & 10 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & -2 & -4 & -6 & 0 \\ 0 & 10 & 20 & 30 & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 2 & 10 & -6 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 3 & 0 & 6 & -12 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From the reduced echelon form, we can see that the system has two free variables, x_4 and x_3. Thus we can write the solution in vector form in terms of $x_4 = t$ and $x_3 = s$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s + 4t \\ -2s - 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 4 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

7. Here we have three vectors in $\mathbb{R}^2$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{v_1} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \quad \mathbf{v_2} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}.$$

The transformation T maps $\mathbf{x}$ onto $x_1\mathbf{v_1} + x_2\mathbf{v_2}$. To find a matrix A such that $T(\mathbf{x})$ is $A\mathbf{x}$ for each $\mathbf{x}$, all need to do is rearrange the given mapping of T

$$x_1\mathbf{v_1} + x_2\mathbf{v_2} = x_1 \begin{bmatrix} -3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 7 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 & 7 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus

$$A = \begin{bmatrix} -3 & 7 \\ 5 & -2 \end{bmatrix}.$$

8. To find the standard matrix of a transformation T that reflects points through the horizontal x_1 axis and then reflects the point through the line $x_2 = x_1$. We determine $T(\mathbf{e_1})$ and $T(\mathbf{e_2})$.
For $\mathbf{e_1}$ reflecting it about x_1-axis yields $\mathbf{e_1}$, and then reflecting the point through the line $x_1 = x_2$, we get $\mathbf{e_2}$.
For $\mathbf{e_2}$ reflecting it about x_1-axis yields $-\mathbf{e_2}$, and then reflecting it about the line $x_1 = x_2$, we get $-\mathbf{e_1}$.
Thus the standard matrix A has the form

$$A = [\,T(\mathbf{e_1}) \quad T(\mathbf{e_2})\,] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

9. Suppose $T(x_1, x_2) = (x_1 + 4x_2, 0, x_1 - 3x_2, x_1)$. We want to find a matrix A such that

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 4x_2 \\ 0 \\ x_1 - 3x_2 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus,

$$A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \\ 1 & -3 \\ 1 & 0 \end{bmatrix}.$$