1. Let $A = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 4 \\ 3 & -2 \end{bmatrix}$. Then $A\mathbf{b_1} = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 12 \\ 3 \end{bmatrix}$,

$A\mathbf{b_2} = \begin{bmatrix} 4 & -3 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 22 \\ -22 \\ -2 \end{bmatrix}$. Thus, $AB = [\,A\mathbf{b_1} \quad A\mathbf{b_2}\,] = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$

We can also compute AB using the row-column rule.

$$AB = \begin{bmatrix} 4(1) - 3(3) & 4(4) - 3(-2) \\ -3(1) + 5(3) & -3(4) + 5(-2) \\ 0(1) + 1(3) & 0(4) + 1(-2) \end{bmatrix} = \begin{bmatrix} -5 & 22 \\ 12 & -22 \\ 3 & -2 \end{bmatrix}$$

2. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then $B \neq C$ and $AB = AC = O_3$.

3. Consider system of equations $7x_1 + 3x_2 = -9, \qquad -6x_1 - 3x_2 = 4$.

(a) We can create a matrix equation of the form $A\mathbf{x} = \mathbf{b}$ where the matrix A is a coefficient matrix and $\mathbf{x}$, $\mathbf{b}$ are vectors in $\mathbb{R}^2$. We have

$$A = \begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mathbf{b} = \begin{bmatrix} -9 \\ 4 \end{bmatrix}.$$

(b) We can compute the inverse of A by the algorithm

$$\begin{bmatrix} 7 & -3 & 1 & 0 \\ -6 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 7 & -3 & 1 & 0 \\ 0 & -3 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 7 & 0 & 7 & 7 \\ 0 & -3 & 6 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & \frac{-7}{3} \end{bmatrix}$$

It follows that $A^{-1} = \begin{bmatrix} 1 & 1 \\ -2 & \frac{-7}{3} \end{bmatrix}$.

(c) We have $\mathbf{x}_0 = A^{-1}\mathbf{b} = \begin{bmatrix} 1 & 1 \\ -2 & \frac{-7}{3} \end{bmatrix} \begin{bmatrix} -9 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ \frac{26}{3} \end{bmatrix}$, and $A\mathbf{x_0} = \mathbf{b}_0$.

4. Let A be a 3×3 matrix. Consider the reduced row echelon form of $[A\ I_3]$.
Note that the columns of the $[A\ I_3]$ span $\mathbb{R}^3$. So, the matrix has three pivoting ones.
Suppose the reduced row echelon form of $[A\ I_3]$ is $[S\ T]$, where both S and T are 3×3.
Then S is the reduced row echelon form of A.

(a) If A has one pivoting one, then S has one of the following form depending on which of the columns of A is the pivoting column: $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Now, the two other pivoting ones of $[A\ I_3]$ must lie in the matrix T. So, T has one of the following form depending on which of 3 columns are the 2 pivoting columns: $\begin{bmatrix} 0 & 0 & x \\ 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}, \begin{bmatrix} 0 & y & 0 \\ 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
Because the columns of I_3 span $\mathbb{R}^3$ and so does T. Thus, x, y, z are nonzero in the above form.
In this case, A is not invertible.

(b) Using a similar analysis as in (a), we see that S has one of the following form depending on which of the columns of A are the pivoting columns: $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, and

T has one of form: $\begin{bmatrix} 0 & * & * \\ 0 & * & * \\ 1 & * & * \end{bmatrix}$, $\begin{bmatrix} * & 0 & * \\ * & 0 & * \\ 0 & 1 & * \end{bmatrix}$, $\begin{bmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$. In this case, A is not invertible.

(c) In this case, $S = I_3$, $T = A^{-1}$, and A is invertible.

5. Suppose A is invertible. If $AB = AC$, then $A(B - C) = O$. Multiply A^{-1} to both sides, we see that $(B - C) = I(B - C) = A^{-1}(A(B - C)) = A^{-1}O = O$. So, $B - C = 0$, i.e., $B = C$.

6 Applying row reduction to the augmented matrix $[\,A \quad I_3\,]$, we get

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & -3 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 2 & -1 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 10 & -2 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 4 & 1 & 0 \\ 0 & 0 & 1 & 10 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 6 & -1 & 0 \\ 0 & 0 & 1 & 10 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 3 & 0 & 1 \\ 0 & 1 & 0 & 6 & -1 & 0 \\ 0 & 0 & 1 & 10 & -2 & 1 \end{bmatrix}.$$

So, $B = \begin{bmatrix} 3 & 0 & 1 \\ 6 & -1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$ and $AB = I_3$.

7. To construct the desired matrix B such that $AB = I_2$, we consider $AB_1 = e_1$ and $AB_2 = e_2$, where $\mathbf{b}_1, \mathbf{b}_2$ are the columns of B, and e_1, e_2 are columns of I_2. We see that

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To construct a matrix C such that $CA = I_4$, the first row of C times A must equal $[\,1 \quad 0 \quad 0 \quad 0\,]$. So

$$[\,c_{11} \quad c_{12}\,] \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = [\,1 \quad 0 \quad 0 \quad 0\,]$$

It follows that

$$c_{11} = 1, \; c_{11} + c_{12} = 0, \; c_{12} = 1.$$

These equations have no solution. So, the desired matrix C does not exist.

8. Suppose the columns of A^2 span $\mathbb{R}^n$. Then A has n pivoting columns, and A^2 has linearly independent columns.

Consider the columns of A. Either (a) they are linearly independent, or (b) they are linearly dependent.

We argue that (b) cannot holds. So, (a) must hold. If (b) holds, then there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = 0$. But then $A^2\mathbf{x} = A(A\mathbf{x})) = A0 = 0$, which is impossible by the given condition on A^2. Therefore, (b) cannot holds.

9. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (2x_1 - 8x_2, -2x_1 + 7x_2)$.

(a) The desired matrix A satisfies $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 - 8x_2 \\ -2x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 2 & -8 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. So, $A = \begin{bmatrix} 2 & -8 \\ -2 & 7 \end{bmatrix}$.

(b) By the determinant method, $A^{-1} = \frac{1}{14-16}\begin{bmatrix} 7 & 8 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{-7}{2} & -4 \\ -1 & -1 \end{bmatrix}$ so that $A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-7}{2}x_1 - 4x_2 \\ -x_1 - x_2 \end{bmatrix}$.

Thus, $R(x_1, x_2) = (\frac{-7}{2}x_1 - 4x_2, -x_1 - x_2)$.

(c) $R \circ T(x_1, x_2) = R(2x_1 - 8x_2, -2x_1 + 7x_2) = (x_1, x_2)$.

(d) $T \circ R(x_1, x_2) = T(\frac{-7}{2}x_1 - 4x_2, -x_1 - x_2) = (x_1, x_2)$.