Math 211 Homework 5 Sample solution

1. Suppose A, B, C are square matrices such that $ABC = I$.

(a) Because $ABC = I$, we see that $A^{-1} = (BC)$. Thus, $I = A^{-1}A = (BC)A = BCA$.

(b) Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Then $ABC = I_2$, but $BAC = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

2. Compute the product of two partitioned matrices

$$\begin{bmatrix} I & 0 \\ -E & I \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} IW + 0Y & IX + 0Z \\ -EW + IY & -EX + IZ \end{bmatrix} = \begin{bmatrix} W & X \\ Y - EW & Z - EX \end{bmatrix}$$

3. Given matrix equation

$$\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}.$$

To find formulas for X, Y, and Z in terms of A and B. Note that

$$\begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} X & Y & Z \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} AX + 0B & AY + 0B & AZ + BI \\ 0X + 0I & 0Y + 0I & 0Z + II \end{bmatrix}$$

$$= \begin{bmatrix} AX & AY & ZA + B \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & I \end{bmatrix}$$

Thus,

$$AX = I, \quad AY = 0, \quad AZ + B = 0.$$

From these equations, we get that

$$X = A^{-1}, \quad Y = A^{-1}(AY) = A^{-1}0 = 0, \quad Z = -A^{-1}B.$$

4. Suppose $A^2 = I_n$. Then $\begin{bmatrix} A & 0 \\ I_n & -A \end{bmatrix} \begin{bmatrix} A & 0 \\ I_n & -A \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ A - A & (-A)(-A) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix}$.

5. Let $A = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22}0 & \\ 0 & 0 & A_{33} \end{bmatrix}$ such that A_{11}, A_{22}, A_{33} are invertible.

(a) Because $\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{bmatrix} A_{11}^{-1} & 0 & 0 \\ 0 & A_{22}^{-1} & 0 \\ 0 & 0 & A_{33}^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$, we see that $A^{-1} = \begin{bmatrix} A_{11}^{-1} & 0 & 0 \\ 0 & A_{22}^{-1} & 0 \\ 0 & 0 & A_{33}^{-1} \end{bmatrix}$.

(b) Suppose $A_{11} = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $A_{22} = \begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix}$, $A_{33} = [2]$. Then $A_{11}^{-1} = -\begin{bmatrix} 5 & -2 \\ -3 & 1 \end{bmatrix}$,

$A_{22}^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}$, $A_{33}^{-1} = [1/2]$ so that $A^{-1} = \begin{bmatrix} -5 & 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & -5/2 & 7/2 & 0 \\ 0 & 0 & 0 & 0 & 1/2 \end{bmatrix}$.

6. Solving $L\mathbf{y} = \mathbf{b}$ by back substitution, we get $\mathbf{y} = \begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix}$. Solving $Ux = y$ by back substitution, we get $\mathbf{x} = \begin{bmatrix} -11 \\ -6 \\ -3 \end{bmatrix}$.

7. To find the LU factorization of the matrix $A = \begin{bmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{bmatrix}$, we apply row reduction to

$$\begin{bmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{bmatrix} \to \begin{bmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 10 & 24 \end{bmatrix} \to \begin{bmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{bmatrix} = U.$$

The two row reduction steps above correspond to multiplying the matrices:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}.$$

So,

$$L = E_1^{-1} E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 5 & 1 \end{bmatrix}.$$

Remark You only need put the negative of the off-diagonal entries of E_1, E_2 to the corresponding positions in L.

8. Suppose B is invertible and $A = BC$. Apply row reduction to $[B|A]$ to get $[I|X]$.
Then there are elementary matrices $E_1, \ldots, E_k$ such that $(E_k \cdots E_1)[B|A] = [I|X]$.
Thus, $(E_k \cdots E_1)B = I$ and $(E_k \cdots E_1)A = X$.
It follows that $(E_k \cdots E_1) = B^{-1}$ and $X = (E_k \cdots E_1)A = B^{-1}A = B^{-1}BC = C$.