Math 211 Homework 6 Sample solution

1. Using the cofactor expansion across the first row, we have

$$\det\left(\begin{bmatrix} 8 & 1 & 6 \\ 4 & 0 & 3 \\ 3 & -2 & 5 \end{bmatrix}\right) = 8\cdot\det\left(\begin{bmatrix} 0 & 3 \\ -2 & 5 \end{bmatrix}\right) - \det\left(\begin{bmatrix} 4 & 3 \\ 3 & 5 \end{bmatrix}\right) + 6\cdot\det\left(\begin{bmatrix} 4 & 0 \\ 3 & -2 \end{bmatrix}\right)$$

$$= 8(6) - 11 + 6(-8) = -\mathbf{11}.$$

2. Using the cofactor expansion, we expand across the top row in each matrix. We have

$$\det\begin{pmatrix} 4 & 0 & 0 & 0 \\ 7 & -1 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 5 & -8 & 4 & -3 \end{pmatrix} = 4\det\begin{pmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{pmatrix} = -4\det\begin{pmatrix} 3 & 0 \\ 4 & -3 \end{pmatrix} = -4(-9) = \mathbf{36}.$$

3. Using the Sarrus rule, we have

$$\det\begin{pmatrix} 1 & 3 & 5 \\ 2 & 1 & 1 \\ 3 & 4 & 2 \end{pmatrix} = 1(1(2)) + 3(1(3)) + 5(2(4)) - 3(1(5)) - 4(1(1)) - 2(2(3))$$

$$= 2 + 9 + 40 - 15 - 4 - 12 = \mathbf{20}.$$

4. Using row reduction, we have

$$\begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & -3 \\ 0 & -18 & 12 \\ 0 & 0 & 1 \end{bmatrix},$$

and no change of the determinant in the process. So, $\det(A) = -\mathbf{18}$.

5. Using row reduction, we have

$$\begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 5 & 4 & -3 \\ -3 & -7 & -5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 2 & 4 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 3 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & -1 & -2 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the determinant of this matrix is 0.

6. Suppose $\det(A) = \det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 7$. To calculate $\det(M) = \det\begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix}$, note that the new matrix N can be derived from the first matrix by exchanging the first and third rows and then switching the new third row with the second row. We can see that $r = 2$ where r is the number of times that the rows were interchanged. As such

$$\det(M) = (-1)^r \det(A) = (-1)^2 \det(A) = 7.$$

The determinant of M is also 7.

To calculate

$$\det(N) = \det \begin{pmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{pmatrix},$$

note that the matrix N is obtained by adding the second row of A to its first row. Because this was a scalar multiple addition, $\det(\mathbf{N}) = \det(\mathbf{A}) = \mathbf{7}$.

7. (a) $\det(rA) = \det(rIA) = \det(rI)\det(A) = r^n \det(A)$.
(b) Note that $1 = \det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$. Thus, $\det(A^{-1}) = 1/\det(A)$.

8. $\det(PAP^{-1}) = \det(P)\det(A)\det(P^{-1}) = \det(P)\det(A)/\det(P) = \det(A)$.

9. Suppose $\det(A) = -1$ and $\det(B) = 2$. Then
(a) $\det(AB) = \det(A)\det(B) = (-1)(2) = -2$,
(b) $\det(B^5) = \det(B)^5 = 2^5$,
(c) $\det(2A) = 2^4 \det(A) = -2^4$,
(d) $\det(AA^T) = \det(A)\det(A^T) = (-1)(-1) = 1$,
(d) $\det(AB^{-1}) = \det(A)\det(B^{-1}) = -1/2$.