Math 211 Homework 7 Sample Solution

1. Use Cramer's rule to compute the solution to this system

$$-5x_1 + 3x_2 = 9, \qquad 3x_1 - x_2 = -5.$$

In matrix form, we have $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} -5 & 3 \\ 3 & -1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}.$$

Let $A_1 = \begin{bmatrix} 9 & 3 \\ -5 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -5 & 9 \\ 3 & -5 \end{bmatrix}$. Then

$$x_1 = \det(A_1)/\det(A) = 6/(-4) = -3/2, \ x_2 = \det(A_2)/\det(A) = -2/(-4) = 1/2.$$

2. Consider the equations

$$2x_1 + x_2 + x_3 = 4, \qquad -x_1 + 3x_3 = 2, \qquad 3x_1 + x_2 + 3x_3 = 2.$$

We can write this in the form $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 3 \\ 3 & 1 & 3 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}.$$

Let A_i be the matrix obtained from A by replacing its ith column with $\mathbf{b}$. Then by Cramer's rule,

$$x_1 = \frac{\det A_1}{\det A} = \frac{-10}{5} = -2, \ x_2 = \frac{\det A_2}{\det A} = \frac{40}{5} = 8, \ x_3 = \frac{\det A_3}{\det A} = \frac{0}{5} = 0.$$

3. Let $A = \begin{bmatrix} 0 & -2 & -1 \\ 3 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}$ Then the adjugate of A is

$$\text{adj}A = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where C_{ij} is the cofactor of the ith row and the jth column. Since $\det(A) = 3$, we have

$$\text{adj}A = \begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix} \quad \text{and} \quad A^{-1} = \frac{\text{adj}A}{\det A} = \frac{1}{3}\begin{bmatrix} 0 & 1 & 0 \\ -3 & -1 & -3 \\ 3 & 2 & 6 \end{bmatrix}$$

4. Compute $(6,-1)-(0,-2) = (6,1), (-3,1)-(0,-2) = (-3,3), (3,2)-(0,-2) = (3,4)$, we see that $(3,4) = (6,1)+(-3,3)$. So, after translation, the parallelogram has vertexes $(0,0), (6,1), (-3,3), (4,3)$ and the area is $\left|\det\left(\begin{bmatrix} 6 & -3 \\ 1 & 3 \end{bmatrix}\right)\right| = 21$.

5. Let

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : xy \geq 0 \right\}.$$

1. For any $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix} \in W$ and any $c \in \mathbb{R}$, we have $xy \geq 0$. Thus, if we multiply the two entries of $c\mathbf{u} = c \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, we get $c^2 u_1 u_2$ which is always nonnegative. Thus, $c\mathbf{u}$ is in W.

2. To find two vectors in W whose sum is not in W, let $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The sum $\mathbf{u}+\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ which is not in W.

6 Let W be all the vectors of the form $\begin{bmatrix} 2s+4t \\ 2s \\ 2s-3t \\ 5t \end{bmatrix} = s\mathbf{v_1} + t\mathbf{v_2}$ with $\mathbf{v_1} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 4 \\ 0 \\ -3 \\ 5 \end{bmatrix}$.

We check the three condition for subspaces.

(a) $\mathbf{0} = 0\mathbf{v_1} + 0\mathbf{v_2} \in W$.

(b) If $\mathbf{x} = s_1\mathbf{v_1} + t_1\mathbf{v_2}$ and $\mathbf{y} = s_2\mathbf{v_1} + t_2\mathbf{v_2}$ are vectors in W, then $\mathbf{x}+\mathbf{y} = (s_1+s_2)\mathbf{v_1} + (t_1 t_2)\mathbf{v_2}$ is a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$. So, $\mathbf{x}+\mathbf{y} \in W$.

(c) If $\mathbf{x} = s_1\mathbf{v_1} + t_1\mathbf{v_2}$ and $c \in \mathbf{R}$, then $c\mathbf{x} = (cs_1)\mathbf{v_1} + (cs_2)\mathbf{v_2} \in W$.

Alternatively, one can use Theorem 1 to conclude that $W = \text{Span}\{\mathbf{v_1}, \mathbf{v_2}\}$ is a subspace.

7. The set W is not a vector space (subspace) because $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is not in the set.

8. Let F be a fixed 3×2 matrix. Let $W = \{X : X \text{ is } 2 \times 4, FX = O_{3\times4}\}$.

(a) $O_{2\times4} \in W$ because $FO_{2\times4} = O_{3\times4}$.

(b) If $U, V \in W$, then $FU = O_{3\times4}$ and $FV = O_{3\times4}$. Hence $F(U+V) = FU+FV = O_{3\times4}+O_{3\times4} = O_{3\times4}$. So, $U+V \in W$.

(c) If $U \in W$ and $c \in \mathbf{R}$, then $F(cU) = cFU = cO_{3\times4} = O_{3\times4}$. So, $cU \in W$.

Combining (a) − (c), we see that W is a subspace.

9. Let H, K be subspace of V. Then

(a) $0 \in H$, (b) $u+v \in H$ if $u, v \in H$, (c) $cu \in H$ if $c \in \mathbf{R}, u \in H$,

(a) $0 \in K$, (b) $u+v \in K$ if $u, v \in K$, (c) $cu \in K$ if $c \in \mathbf{R}, u \in K$.

Consider $H+K = \{h+k : x \in H, y \in K\}$.

(a) $0 = h+k$ with $h = 0 \in H, k = 0 \in K$. So, $0 \in H+K$.

(b) If $x, y \in H_K$, then $x = h_1+k_1, y = h_2+k_2$ with $h_1, h_2 \in H, y_1, y_2 \in K$. Then $x+y = h+k$ with $h = h_1+h_2 \in H$ and $k = k_1+k_2 \in K$. So, $x+y \in H+K$.

(c) If $c \in \mathbf{R}$ and $x = h+k \in H+K$ with $h \in H$ and $k \in K$, then $cx = ch+ck \in H+K$ because $ch \in H, ck \in K$.

Combining (a) − (c), we see that $H+K$ is a subspace.