Math 211 Homework 8 Sample solution

1. Let

$$A = \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

Nul$A$ is the set of all vectors $\mathbf{x}$ such that $A\mathbf{x} = 0$. Apply row reduction to the augmented matrix, we have

$$\begin{bmatrix} 1 & -3 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

From this, we can see that

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \\ 0 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, Nul$A$ is the set of linear combinations of these two vectors, i.e., they Span Nul$A$.

2. Let

$$A = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}, \qquad \mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

To determine if $\mathbf{w}$ is in Col$A$. By definition, Col$A$ is the set of all vectors $\mathbf{b}$ such that $\mathbf{b} = A\mathbf{x}$.
Let $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}$. We have $A\mathbf{x} = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \mathbf{w}$. So $\mathbf{w}$ is in Col$A$.
To determine if $\mathbf{w}$ is in Nul$A$, we multiply the matrices and study the result.

$$A\mathbf{w} = \begin{bmatrix} 10 & -8 & -2 & -2 \\ 0 & 2 & 2 & -2 \\ 1 & -1 & 6 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So $\mathbf{w}$ is in Nul A.

3. Let $T : \mathbb{P}_2(t) \to \mathbb{R}^2$ be defined by $T(\mathbf{p}) = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(0) \end{bmatrix}$. Then $T(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix}$. Thus, $T(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_0 \\ a_0 \end{bmatrix}$ means $a_0 = 0$. So, the kernel of T is the set $\{a_1 t + a_2 t^2 : a_1, a_2 \in \mathbb{R}\}$, which is the span of the two polynomials: $p(t) = t, q(t) = t^2$.
The range of T is the set $\{T(a_0 + a_1 t + a_2 t^2) : a_0, a_1, a_2 \in \mathbb{R}\} = \left\{ \begin{bmatrix} a_0 \\ a_0 \end{bmatrix} : a_0 \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

4. Apply row reduction to $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$ to get $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So, the column space is spanned by the pivoting columns of A, i.e., columns 1 and 2. The null space consists of vectors of the form: $x_3\mathbf{v}_1 + x_4\mathbf{v}_2$ and equals the space of the vectors $\mathbf{v}_1, \mathbf{v}_2$ with

$$\mathbf{v}_1 = \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}.$$

5. To find a basis for the space spanned by thefive vectors, study the space spanned by these vectors is Col$A$ where A is the matrix with these vectors as columns. Applying row reduction to A to get

$$\begin{bmatrix} 1 & -2 & 3 & 5 & 2 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 1 & 2 & -1 & -4 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 3 & 5 & 2 \\ 0 & 4 & -4 & -9 & -2 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we see that the first three columns are the pivot columns. So a basis for this space consists of the first three vectors.

6. Note that the first two vectors are linearly independent because none is the multiple of the other. Because $\mathbf{v_3} = 2\mathbf{v}_1, -\mathbf{v}_2$ is a linear combination of the first two vectors, the first two vectors form a basis for this space.

7. Note that $\cos t$ is not the zero function. So, $\{\cos t\}$ is linearly independent. Now, $\sin t$ is a not multiple of $\cos t$. So, $\{\cos t, \sin t\}$ is linearly independent. We will show that if $\sin 2t = c_1 \cos t + c_2 \sin t$, then $c_1 = c_2 = 0$. To see this, consider $t = 0$. We get the equation $0 = c_1 + 0$. So, $c_1 = 0$. Then let $t = \pi/2$, we get the equation $0 = c_2$. So, $c_2 = 0$. Hence $\{\cos t, \sin t, \sin 2t\}$ is linearly independent. Now, $\sin t \cos t = 0 \cos t + 0 \sin t + (1/2) \sin 2t$. So, we discard $\sin t \cot t$, and conclude that $\{\cos t, \sin t, \sin 2t\}$ is a basis for H.