1. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix} \right\}$$

be a basis for $\mathbb{R}^3$. (a) Find $\mathbf{x}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$. (b) Find $[\mathbf{y}]_{\mathcal{B}}$ if $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Solution. (a) We have

$$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 & 3 & 4 \\ 2 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ -5 \\ 1 \end{bmatrix}.$$

(b) We have

$$[\mathbf{y}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{y} = \begin{bmatrix} 3/2 \\ 0 \\ 1 \end{bmatrix}.$$

2. Let $\mathcal{B}$ be the basis in Question 1, and

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

be another basis basis for $\mathbb{R}^3$.

(a) Find the matrix $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$. (b) Find $[\mathbf{z}]_{\mathcal{C}}$ if $[\mathbf{z}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution. (a) $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \mathcal{P}_{\mathcal{C}}^{-1}\mathcal{P}_{\mathcal{B}}$ equals

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 3 & 4 \\ 2 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 4 \\ 2 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 5 \\ 2 & -2 & -4 \\ 0 & 2 & 3 \end{bmatrix}.$$

(b) $[\mathbf{z}]_{\mathcal{C}} = \mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{z}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -4 \\ 5 \end{bmatrix}$.

3. Suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{R}^3$. Show that $\mathcal{C} = \{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_1 - \mathbf{b}_3\}$ is also a basis, and find the matrix $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}}$.

Solution. We show that $\mathcal{C}$ is linearly independent. Suppose

$$\mathbf{0} = \gamma_1\mathbf{b}_1 + \gamma_2(\mathbf{b}_1 + \mathbf{b}_2) + \gamma_3(\mathbf{b}_1 - \mathbf{b}_3) = (\gamma_1 + \gamma_2 + \gamma_3)\mathbf{b}_1 + \gamma_2\mathbf{b}_2 - \gamma_3\mathbf{b}_3.$$

Because $\mathcal{B}$ is linearly independent, we see that

$$\gamma_1 + \gamma_2 + \gamma_3 = 0, \qquad \gamma_2 = 0, \qquad -\gamma_3 = 0.$$

So, $\gamma_1 = \gamma_2 = \gamma_3 = 0$. It follows that $\mathcal{C}$ is a basis for $\mathbf{R}^3$.

Note that $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}}^{-1}$, and the columns of $\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}}$ are $[\mathbf{c}_1]_{\mathcal{B}}, [\mathbf{c}_2]_{\mathcal{B}}, [\mathbf{c}_3]_{\mathcal{B}}$. So,

$$\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

4. Find the dimension of the subspace H of $\mathbb{R}^3$ spanned by $\begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}$.

Solution. Apply row reduction:

$$\begin{bmatrix} 1 & -3 & -2 & -3 \\ -2 & -6 & 3 & 5 \\ 0 & 0 & 5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -2 & -3 \\ 0 & 12 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

So, dimension of $H = 3$, i.e., $H = \mathbb{R}^3$

5. Find the dimensions of Nul A and Col A for the matrix

$$A = \begin{bmatrix} 1 & 2 & -4 & 3 & -2 & 6 & 0 \\ 0 & 0 & 0 & 1 & 0 & -3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Determine rank$A$, $\dim \text{Nul}A$, and a basis for the row space of A.

Solution. Rank$A = 4$, $\dim \text{Nul}A = 7 - 4 = 3$, the four rows of A will form a basis for the row space of A.

6. Let A be a 7×5 matrix with rank 2. Determine $\dim \text{Nul}A$ and rankA^T.

Solution. $\dim \text{Nul}A = 5 - 2 = 3$ and rankA^T = rank $A = 2$.

7. Consider the linear system $Ax = b$ such that A is 6×8. Suppose A has rank 6.

(a) Is there any b such that the system is inconsistent?

(b) If there any b such that the system has a unique solution?

Soluton. (a) Since A has rank 6, the column space of A equals $\mathbb{R}^6$. So, every $A\mathbf{x} = \mathbf{b}$ has solutions for any $\mathbf{b} \in \mathbb{R}^6$.

(b) There are always 2 free variables, so there are always infinitely many solutions.

8. Let $H = \{(a, b, c, d) : a - 3b + c = 0\}$.

(a) Show that H is a subspace of $\mathbb{R}^{1\times 4}$.

(b) Find a basis for H, and hence deduce the dimension of H.

Solution. (a) Note that (a.1) $(0, 0, 0, 0) \in H$, (a.2) If $u = (a_1, b_1, c_1, d_1), v = (a_2, b_2, c_2, d_2) \in H$, then $a_1 - 3b_1 + c_1 = 0$ and $a_2 - 3b_2 + c_2 = 0$. So, $u + v = (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2) \in H$ because $(a_1 + a_2) - 3(b_1 + b_2) + (c_1 + c_2) = 0$. (a.3) for any real number γ, $\gamma u = (\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1) \in H$ because $\gamma a_1 - 3\gamma b_1 + \gamma c_1 = \gamma(a_1 - 3b_1 + c_1) = 0$.

(b) Note that $(a, b, c, d) \in H$ means that $[1 \ \ -3 \ \ 1 \ \ 0] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$. Solving the equation, we see that $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. So, $H = \text{span}\{(3, 1, 0, 0), (-1, 0, 1, 0), (0, 0, 0, 1)\}$. Now, $\{(3, 1, 0, 0), (-1, 0, 1, 0), (0, 0, 0, 1)\}$ is linearly independent; so it is a basis for H.

9. Let $W = \{a + bt + ct^2 + dt^3 : a - 3b + c = 0\}$.

(a) Show that W is a subspace of $\mathbb{P}_3(t)$.

(b) Find a basis for W.

Solution. (a) Note that (a.1) $0 + 0t + 0t^2 + 0t^3 \in W$, (a.2) If $u = a_1 + b_1t + c_1t^2 + d_1t^3, v = a_2 + b_2t + c_2t^2 + d_2t^3 \in W$, then $a_1 - 3b_1 + c_1 = 0$ and $a_2 - 3b_2 + c_2 = 0$. So, $u + v = (a_1 + a_2) + (b_1 + b_2)t + (c_1 + c_2)t^2 + (d_1 + d_2)t^3 \in W$ because $(a_1 + a_2) - 3(b_1 + b_2) + (c_1 + c_2) = 0$. (a.3) for any real number γ, $\gamma u = \gamma a_1 + \gamma b_1 t + \gamma c_1 t^2 \gamma d_1 t^3 \in W$ because $\gamma a_1 - 3\gamma b_1 + \gamma c_1 = \gamma(a_1 - 3b_1 + c_1) = 0$.

(b) Note that $a + bt + ct^2 + dt^3 \in W$ means that $[1 \ -3 \ 1 \ 0] \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$. Solving the equation, we see that $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. So, $W = \text{span}\{3 + t, -1 + t^2, t^3\}$. Now, $\{3 + t, -1 + t^2, t^3\}$ is linearly independent; so it is a basis for W.

Remark The calculation of Questions 8 and 9 are essentially the same.