Math 211 Homework 10 Sample solution

1. (10 points) Show that

$$\mathcal{B} = \{1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3\}$$

is a basis for $\mathbb{P}_3$, and find the change of basis matrix from $\mathcal{B}$ to $\mathcal{C} = \{1, t, t^2, t^3\}$ and the change of the basis matrix from $\mathcal{C}$ to $\mathcal{B}$ Find $[u]_\mathcal{B}$ for $u = 1-3t+9t^2-t^3$.

Solution. Suppose $0 = \gamma_1 1 + \gamma_2(1-t) + \gamma_3(2-4t+t^2) + \gamma_4(6-18t+9t^2-t^3)$. Then

$$B\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ 4\gamma_4 \end{bmatrix} = \mathbf{0} \quad \text{with} \quad B = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

So, $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = 0$. Thus, $\mathcal{B}$ is a basis. Now, $\mathcal{P}_{\mathcal{C}\leftarrow\mathcal{B}} = B$ so that $\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{C}} = B^{-1} = B$. (Just check that $BB = I_4$.) So, $[u]_\mathcal{B} = B[u]_\mathcal{C} = B\begin{bmatrix} 1 \\ -3 \\ 9 \\ -1 \end{bmatrix} = \begin{bmatrix} 10 \\ -15 \\ 0 \\ 1 \end{bmatrix}$.

2. (5 points) Find the rank, the nullity (dimension of null space), a basis for the column space, a basis for the row space of the following matrix

$$\begin{bmatrix} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & -13 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Solution. The rank is 5, which is the number of leading ones; the nullity is $6-5=1$, Columns 1, 2, 3, 5, 6 are pivoting columns, and they will form a basis for the column space; Rows 1, 2, 3, 4, 5 form a basis for the row space.

3. Find the characteristic polynomial for the matrix and two linearly independent eigenvectors for the matrix

$$A = \begin{bmatrix} 4 & 4 \\ 1 & 4 \end{bmatrix}.$$

Solution. The characteristic polynomial is $\det(A-\lambda I) = (4-\lambda)(4-\lambda) - 4 = \lambda^2 - 8\lambda + 12 = (\lambda-2)(\lambda-6)$. We see the eigenvalues are $2, 6$. Solving $(A-2I)x = 0$ and $(A-6I)x = 0$, we get the following linearly independent eigenvectors: $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

4. Show that $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ has at most one linear independent eigenvector.

Solution. Note that A has only one eigenvalue, namely, $\lambda = 1$. Now, $A - I$ has rank 2. So, the null space has dimension 1. So, there is only one linearly independent eigenvector.

5. Find the characteristic polynomial for the matrix and three linearly independent eigenvectors for the matrix

$$A = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. The characteristic polynomial is $\det(A-\lambda I) = (4-\lambda)(3-\lambda)(2-\lambda)$. So, the eigenvalues are $4, 3, 2$. Solving the systems $(A-4I)x = 0$, $(A-3I)x = 0$ and $(A-2I)$, we get the following linearly independent eigenvectors:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ -8 \\ 2 \end{bmatrix}.$$

solving $(A-3I)x = 0$, we get an eigenvector e_2. **6.** Find the values h so that there are two linearly independent eigenvectors corresponding to the eigenvalue $\lambda = 4$ for the matrix

$$A = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Solution. Apply row reduction to $A - 4I$ to get

$$\begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & h+3 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus, the matrix has nullity 2 when $h+3=0$, i.e., $h=-3$.

7. Show that A and A^T have the same characteristic polynomial.

Solution. Note that $\det(X) = \det(X^T)$ and $(X-Y)^T = X^T - Y^T$. Thus

$$\det(A-\lambda I) = \det((A-\lambda I)^T) = \det(A^T - (\lambda I)^T) = \det(A^T - \lambda I).$$