Math 211 Sample solution of Homework 11

1. Given that $\lambda = 2, 3$ are eigenvalues for the matrix

$$A = \begin{bmatrix} 2 & 0 & -2 \\ 1 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Find an invertible matrix P and a diagonal matrix D such that $A = P^{-1}DP$.

(b) Show that $(D-2I)(D-3I) = 0$ and $(A-2I)(A-3I) = 0$.

Solution. (a) Solving $(A-2I)\mathbf{x} = 0$, we get an eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$; solving $(A-3I)\mathbf{x} = 0$, we get two linearly independent eigenvectors: $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Let $V = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$.

Then $AV = VD$ with $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Since P is invertible, $A = VDV^{-1}$. Letting $P = V^{-1}$, we have $A = P^{-1}DP$.

(b) Note that $(D-2I)(D-3I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$. So,

$(A-2I)(A-3I) = (P^{-1}DP-2I)(P^{-1}DP-3I) = P^{-1}(D-2I)(D-3I)P = P^{-1}0P = 0.$

2. GIven a 7×7 matrix A has three different eigenvalues $\lambda_1, \lambda_2, \lambda_3$.
Suppose the $\text{Nul}(A - \lambda_1 I)$ is three-dimensional and $\text{Nul}(A - \lambda_2 I)$ is two-dimensional.

(a) If A is diagonalizable, what can we say about the dimension of $\text{Nul}(A - \lambda_3 I)$?

(b) If A is not diagonalizable, what can we say about the dimension of $\text{Nul}(A - \lambda_3 I)$?

Solution. (a) If A is diagonalizable, we need 7 independent eigenvectors. So, there must be two linearly independent eigenvectors for λ_3. Hence, $\text{Nul}(A - \lambda_3 I)$ has dimension 2.

(b) If A is not diagonalizable, then there is at most one linearly independent eigenvector for λ_3. Hence, $\text{Nul}(A - \lambda_3 I)$ has dimension 1.

3. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be the bases for vector spaces W and V respectively. Let $T : W \to V$ with

$$T(\mathbf{d}_1) = 3\mathbf{b}_1 - 3\mathbf{b}_2 \text{ and } T(\mathbf{d}_2) = -2\mathbf{b}_1 + 5\mathbf{b}_2.$$

Find the matrix for T relative to $\mathcal{D}$ and $\mathcal{B}$.

Solution. The matrix M is 2×2 with first column $[T(\mathbf{d}_1)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$ and second column $[T(\mathbf{d}_2)]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$. Thus, $M = \begin{bmatrix} 3 & -2 \\ -3 & 5 \end{bmatrix}$.

4. Let $T : \mathbb{P}_2 \to \mathbb{P}_4$ be such that $T(\mathbf{p}(t)) = \mathbf{p}(t) + 2t^2\mathbf{p}(t)$.

(a) Find the image of $\mathbf{p}(t) = 3 - 2t + t^2$.

(b) Show that T is a linear transformation.

(c) Find a matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3, t^4\}$.

Solution. (a) $T(3-2t+t^2) = (3-2t+t^2)+2t^2(3-2t+t^2) = 3-2t+7t^2-4t^3+2t^4$.
(b) For any $\mathbf{p}(t), \mathbf{q}(t) \in \mathbb{P}_2$ and $c \in \mathbf{R}$,

$$\begin{aligned} T(\mathbf{p}(t)+\mathbf{q}(t)) &= (\mathbf{p}(t)+\mathbf{q}(t))+2t^2(\mathbf{p}(t)+\mathbf{q}(t)) \\ &= (\mathbf{p}(t)+2t^2\mathbf{p}(t))+(\mathbf{q}(t)+2t^2\mathbf{q}(t)) \\ &= T(\mathbf{p}(t))+T(\mathbf{q}(t)) \end{aligned}$$

and

$$T(c\mathbf{p}(t)) = (c\mathbf{p}(t))+2t^2(c\mathbf{p}(t)) = c(\mathbf{p}(t)+2t^2\mathbf{p}(t)) = c(T(\mathbf{p}(t)).$$

So, T is linear.

(c) The matrix M of T with respect to the canonical bases $\mathcal{C}_1 = \{1, t, t^2\}$ and $\mathcal{C}_2 = \{1, t, t^2, t^3, t^4\}$ is 5×3 with the columns

$$[T(1)]_{\mathcal{C}_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \ [T(t)]_{\mathcal{C}_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \ [T(t^2)]_{\mathcal{C}_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}. \quad \text{Thus } M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

5. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be a basis for a vector space V. Suppose $T : V \to V$ is a linear transformation whose matrix relative to the basis $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix}.$$

Find $T(4\mathbf{b}_1 - 3\mathbf{b}_2)$.
Solution. Let $x = 4b_1 - 3b_2$. Then

$$[T(x)]_{\mathcal{B}} = [T]_{\mathcal{B}}[x]_{\mathcal{B}} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & -2 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -5 \end{bmatrix}.$$

So, $T(x) = 5\mathbf{b}_2 - 5\mathbf{b}_3$.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.
Find a basis $\mathcal{B}$ with the property that $[T]_{\mathcal{B}}$ is diagonal.
Solution. Solving $0 = \det(A - \lambda I) = (2-\lambda)^2 - 9 = (\lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1)$, we see that the eigenvalues are $5, -1$. Solving $(A - 5I)\mathbf{x} = 0$ and $(A + I)\mathbf{x} = 0$, we get the eigenvectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. If $P = [\mathbf{b}_1 \ \mathbf{b}_2]$, then $P^{-1}AP = D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$.
Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Then $[T]_{\mathcal{B}} = D$.

7. Let $T : \mathbb{P}_1(t) \to \mathbb{P}_1(t)$ be defined by $T(a + bt) = (2a + 3b) + (3a + 2b)t$.
Find a basis $\mathcal{B}$ of $\mathbb{P}_1(t)$ with the property that $[T]_{\mathcal{B}}$ is diagonal.
Solution. Using the canonical basis $\mathcal{C} = \{1, t\}$, we have $[T]_{\mathcal{C}} = A$ in question 6.
We know that $P^{-1}AP = D$ as in question 6. Let $P = \mathcal{P}_{\mathcal{C} \leftarrow \mathcal{B}}$. Then $\mathcal{B} = \{\mathbf{p}(t), \mathbf{q}(t)\}$ will the desired basis, and $[\mathbf{p}(t)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $[\mathbf{q}(t)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. So, $\mathcal{B} = \{1 + t, 1 - t\}$.