Math 211 Homework 12 Sample solution

1. Compute $\left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x}$, where

$$\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix}.$$

Solution. $\mathbf{x}\cdot\mathbf{w} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ -5 \end{bmatrix} = 18+2-15 = 5, \quad \mathbf{x}\cdot\mathbf{x} = \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix} = 36+4+9 = 49,$

and hence $\left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{x}\cdot\mathbf{x}}\right)\mathbf{x} = \frac{5}{49}\begin{bmatrix} 6 \\ -2 \\ 3 \end{bmatrix}$. .

2. Find a unit vector in the direction of the vector $\mathbf{x} = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$.

Solution. $||\mathbf{x}|| = \sqrt{36+16+9} = \sqrt{61}$; the unit vector in the direction of $\mathbf{x}$ equals $\frac{1}{\sqrt{61}}\begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$.

3. Let $\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$.

Check which pairs of the above vectors are orthogonal.

Solution. $\mathbf{u}, \mathbf{v}$ are orthogonal because $\mathbf{u}\cdot\mathbf{v} = 24 - 9 - 15 = 0$; $\mathbf{u}, \mathbf{w}$ are not orthogonal because $\mathbf{u}\cdot\mathbf{w} = 6+9+3 \neq 0$, $\mathbf{v}, \mathbf{w}$ are orthogonal because $\mathbf{v}\cdot\mathbf{w} = 2+2-4 = 0$.

4. Consider the set of vectors

$$\left\{ \begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

(a) Show that it is an orthogonal set.

(b) Find a nonzero vector orthogonal to the span of the set.

Solution. (a) The first two vectors are orthogonal to the zero vector, which is the third vector. One only needs to show that the first two vectors are orthogonal by checking

$$\begin{bmatrix} 2 \\ -5 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix} = 8 + 10 - 18 = 0.$$

(b) Solve $Ax = 0$ for $A = \begin{bmatrix} 2 & -5 & -3 \\ 4 & -2 & 6 \end{bmatrix}$. We get the vector $\begin{bmatrix} -9 \\ -6 \\ 4 \end{bmatrix}$.

5. Show that $\{\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}\}$ is an orthogonal basis for $\mathbb{R}^3$ if

$$\mathbf{u_1} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}, \quad \mathbf{u_3} = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}.$$

Express $\mathbf{x} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$.

Solution. Checking

$$\mathbf{u_1} \cdot \mathbf{u_2} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} = 6 - 6 + 0 = 0$$

$$\mathbf{u_1} \cdot \mathbf{u_3} = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 3 - 3 + 0 = 0, \quad \mathbf{u_2} \cdot \mathbf{u_3} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} = 2 + 2 - 4 = 0,$$

we see that the given set is orthogonal.

We compute $c_1 = \frac{\mathbf{u_1} \cdot \mathbf{x}}{\mathbf{u_1} \cdot \mathbf{u_1}} = \frac{15+9+0}{9+9+0} = \frac{4}{3}$, $\quad c_2 = \frac{\mathbf{u_2} \cdot \mathbf{x}}{\mathbf{u_2} \cdot \mathbf{u_2}} = \frac{10-6-1}{4+4+1} = \frac{1}{3}$, $\quad c_3 = \frac{\mathbf{u_3} \cdot \mathbf{x}}{\mathbf{u_3} \cdot \mathbf{u_3}} = \frac{5-3+4}{1+1+16} = \frac{1}{3}$.

Thus $\mathbf{x} = \frac{4}{3}\mathbf{u}_1 + \frac{1}{3}\mathbf{u}_2 + \frac{1}{3}\mathbf{u}_3$.

6. Let $\mathbf{y} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$. Express $\mathbf{y} = a\mathbf{u} + b\mathbf{z}$ such that $\mathbf{z}$ is orthogonal to $\mathbf{u}$.

Solution. We can find $\hat{\mathbf{y}} = \alpha \mathbf{y}$ where $\alpha = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} = \frac{14+6}{49+1} = \frac{2}{5}$. So $\hat{\mathbf{y}} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$. We can let $b = 1$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$. Thus, $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z} = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$.

7. Let $\mathbf{y} = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find the distance from $\mathbf{y}$ to the line passing through $\mathbf{u}$.

Solution. Note that $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}} = (\mathbf{y} \cdot \mathbf{u})/(\mathbf{u} \cdot \mathbf{u})\mathbf{u} = 3\mathbf{u} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. So, $\mathbf{z} = \mathbf{y} - 3\mathbf{u} = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ and the distance required is $\|\mathbf{z}\| = \sqrt{36 + 9} = \sqrt{45}$.

8. Let $\mathbf{x}_1 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1/3 \\ h \\ 0 \end{bmatrix}$. Determine h so that the two vectors are orthogonal; then find positive numbers a, b so that $a\mathbf{x}_1$ and $b\mathbf{x}_2$ are orthonormal.

Solution. $\mathbf{x}_1 \cdot \mathbf{x}_2 = (-2 + 3h)/9$. So, the two vectors are orthogonal if $h = 2/3$.

Now, $\mathbf{x}_1/\|\mathbf{x}_1\| = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}/3$ and $\mathbf{x}_2/\|\mathbf{x}_2\| = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}/\sqrt{5}$ are unit vectors.

So, we can take $a = 1, b = 3/\sqrt{5}$.