Math 211 Homework 13 Sample Solution

1. Let

$$\mathbf{y} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix}.$$

(a) To show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set, note that $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$.

(b) Express $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ so that $\hat{\mathbf{y}}$ is orthogonal projection of the vector $\mathbf{y}$ onto $W =$Span$\{\mathbf{u}_1, \mathbf{u}_2\}$.

We have $\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = 2\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2 = \begin{bmatrix} 3/2 \\ 7/2 \\ 1 \end{bmatrix}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -5/2 \\ 1/2 \\ 2 \end{bmatrix}$.

(c) Find the distance of $\mathbf{y}$ to the subspace W.

The distance is $\|\mathbf{z}\| = \sqrt{42}/2$.

(d) Normalize the vectors in $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}\}$ to get orthonormal basis.

The orthonormal basis is $\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{14}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{42}}\begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix} \right\}$.

2. Let

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix} \quad \text{and } \mathbf{u}_2 = \begin{bmatrix} -4 \\ 3 \\ 1 \\ 0 \end{bmatrix}.$$

(a) Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal set.

Note that $\mathbf{u}_1 \cdot \mathbf{u} = -12 + 12 = 0$.

(b) Express $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ so that $\hat{\mathbf{y}}$ is orthogonal projection of the vector $\mathbf{y} = \begin{bmatrix} 6 \\ 3 \\ -2 \\ 2 \end{bmatrix}$ onto

$W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$.

We have $\hat{\mathbf{y}} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{16}{13}\mathbf{u}_1 + \frac{-17}{26}\mathbf{u}_2 = \begin{bmatrix} 82/13 \\ 77/26 \\ -17/26 \\ 16/13 \end{bmatrix}$ and $\mathbf{z} = \mathbf{y} - \hat{\mathbf{y}} = \begin{bmatrix} -4/13 \\ 1/26 \\ -35/26 \\ 10/13 \end{bmatrix}$.

(c) Find the distance of $\mathbf{y}$ to the subspace W.

The distance is $\|z\| = \sqrt{10}/2$.

(d) Find a nonzero vector $\mathbf{z}_2$ so that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{z}, \mathbf{z}_2\}$ is an orthogonal basis for $\mathbf{R}^4$.

Solving the system with coefficient matrix $\begin{bmatrix} 3 & 4 & 0 & 1 \\ -4 & 3 & 1 & 0 \\ -8 & 1 & -35 & 20 \end{bmatrix}$, we obtain $\mathbf{z}_2 = \begin{bmatrix} -1 \\ -8 \\ 20 \\ 35 \end{bmatrix}$.

(e) Normalize the vectors in S to get an orthonormal basis.

The orthonormal basis is $\left\{ \frac{1}{\sqrt{26}}\begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{26}}\begin{bmatrix} -4/13 \\ 1/26 \\ -35/26 \\ 10/13 \end{bmatrix}, \frac{1}{13\sqrt{10}}\begin{bmatrix} -8 \\ 1 \\ -35 \\ 20 \end{bmatrix}, \frac{1}{13\sqrt{10}}\begin{bmatrix} -1 \\ -8 \\ 20 \\ 35 \end{bmatrix} \right\}$

3. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ 14 \\ -7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(a) Show that $\mathcal{B}$ is a basis for $\mathbf{R}^3$.

Consider $A = \begin{bmatrix} 3 & -3 & 0 \\ -4 & 14 & 1 \\ 5 & -7 & 0 \end{bmatrix}$. We have $\det(A) = -(3)(14) + (5)(-3) \neq 0$. So, A is invertible and has linearly independent columns.

(b) Apply the Gram-Schmidt process to $\mathcal{B}$ to get an orthogonal basis $\mathcal{B}_1$.

By routine calculation, we have $\mathcal{B}_1 = \left\{ \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -7 \\ 1 \\ 5 \end{bmatrix}, \right\}$.

(c) Normalize the vectors in $\mathcal{B}_1$ to get and orthnormal basis $\mathcal{B}_2$.

By routine calculation, we have $\mathcal{B}_2 = \left\{ \frac{1}{5\sqrt{2}} \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{5\sqrt{3}} \begin{bmatrix} -7 \\ 1 \\ 5 \end{bmatrix} \right\}$.

(d) Let A be the matrix with the vectors in $\mathcal{B}$ as columns, and let V be the matrix with the vectors in $\mathcal{B}_2$ as columns. Show that $V^T A = R$ is in upper triangular from.

Direction computation yields $R = V^T A = \begin{bmatrix} 10/\sqrt{2} & -20/\sqrt{2} & -4/5\sqrt{2} \\ 0 & 18/\sqrt{6} & 2/\sqrt{6} \\ 0 & 0 & 1/5\sqrt{3} \end{bmatrix}$.

4. Let

$$A = \begin{bmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{bmatrix}.$$

(a) Show that A has linearly independent columns.

Note that the first 3 rows of A are linearly independent. So, A has rank 3, and has linearly independent columns.

(b) Apply the Gram-Schmidt process to the columns of A to get an orthogonal basis $\mathcal{B}$ for the column space of A.

Direct computation yields $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

(c) Normalize the vectors in $\mathcal{B}$ to get an orthonormal basis $\mathcal{U}$ for the column basis for A.

Direct computation yields $\mathcal{B} = \left\{ \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{8}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \right\}$.

(d) Let Q be the matrix with the vectors in $\mathcal{U}$ as columns. Show that $Q^T A = R$ is in upper triangular form.

Direct computation yields $R = Q^T A = \begin{bmatrix} 2 & 8 & 7 \\ 0 & 2\sqrt{2} & 3\sqrt{2} \\ 0 & 0 & 6 \end{bmatrix}$

5. Repeat the procedures in Question 4 for the matrix $A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}$ to find the QR factorization of A.

Solution. Let $\mathbf{a}_1, \mathbf{a}_2$ be the columns of A. Then $\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} -2 \\ 5 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1} \mathbf{a}_1 = \begin{bmatrix} 5 \\ 2 \\ -4 \\ 2 \end{bmatrix}$.

Normalize these vector yields $Q = \frac{1}{7} \begin{bmatrix} -2 & 5 \\ 5 & 2 \\ 2 & -4 \\ 4 & 2 \end{bmatrix}$.

We have $R = Q^T A = 7 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.