**Math 211** **Exam 1** **Name:**________________________

**10 points for each question. Show your steps clearly. Good luck!**

1. Determine $h, k$ so that the system of equation with augmented matrix $\begin{bmatrix} 1 & 3 & k \\ 4 & h & 8 \end{bmatrix}$ has

   (i) no solution, (ii) 1 solution, (iii) infinitely many solutions.

2. Let $S_1 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = 2x \right\}$, and $S_2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = x/2 \right\}$.

   (a) Give an example of a vector in $S_1$ and a vector in $S_2$.

   (b) Write the vector $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$ as the sum of $v_1$ and $v_2$ with $v_1 \in S_1$ and $v_2 \in S_2$.

3. Determine whether the following sets are linearly dependent with explanation.

   (a) $\{v_1, v_2, v_3\}$ in $\mathbf{R}^2$.

   (b) $\{v_1, v_2, v_3\}$ in $\mathbf{R}^4$ with $v_3$ equal to the zero vector.

4. Consider the linear transformation $T$ such that $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} 3 & 1 \\ 5 & 7 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

   (a) Give an example of $y \in \mathbf{R}^3$ such that $T(x) \neq y$ for any $x \in \mathbf{R}^2$.

   (b) Show that $T$ is one-to-one, i.e., if $u$ and $v$ in $\mathbf{R}^2$ satisfy $T(u) = T(v)$, then $u = v$.

5. (a) Give an example of a $2 \times 2$ matrix $A$ such that $AA^T \neq A^T A$.

   (b) Give an example of three $2 \times 2$ matrices $X, Y, Z$ such that $X$ and $Y$ are invertible matrices, and $Z$ is nonzero such that $XZ = YZ$.

6. Find the $LU$ factorization of the following matrix: $A = \begin{bmatrix} 1 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{bmatrix}$.

7. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$, AND find the inverses $A^T$, and $2A$.

[Hint: Note that $(A^T)^{-1} = (A^{-1})^T$ and $(2A)^{-1} = ((2I)A)^{-1} = \frac{1}{2}A^{-1}$.]

8. Suppose $A = [v_1|v_2|v_3]$ is an invertible $3 \times 3$ matrix, where $v_1, v_2, v_3 \in \mathbf{R}^3$ are the columns of $A$, and $Ax = b$ with $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

(a) Show that $A^{-1}v_1 = e_1, A^{-1}v_2 = e_2, A^{-1}v_3 = e_3$, where $e_1, e_2, e_3$ are the three columns of $I_3$.

(b) Suppose $Ax = b$, and $B$ is obtained from $A$ by replacing its second column by $b$.
Show that $A^{-1}B = [A^{-1}a_1 \ A^{-1}b \ A^{-1}a_3] = [e_1 \ x \ e_3]$, AND deduce that $\det(A^{-1}B) = x_2$.