Linear Algebra: Section 1.1 & 1.2

Objective: Identify and learn how to solve linear systems

Remark Solving (large) linear systems is very important in many branches of study and research. Read the introduction of Chapter 1, 1.6, 1.10.

Concepts:

- A linear equation, a linear system - coefficients, variables.
  Example: $\sqrt{3}x_1 + 4x_2 - ex_3 = \pi$.
  General form: $a_1x_1 + \cdots + a_nx_n = b_n$.
  Example of non-linear equations: $x^2 + x + 2 = 0$, $\sin x + \cos x = 1$.
- Examples and graphical representation in $\mathbb{R}^2$.
- Basic questions: find a solution/all solutions if it exists/they exist.
- A system is consistent if there is at least one solution; else it is inconsistent.

Examples

Chicken and rabbits problem

Question In a farm (of McDonald?), there are x_1 chicken and x_2 rabbits. A head count and leg count of the animals are 20 and 50, respectively. How many chicken and rabbits are there?

Solution Set up the (linear) equations: $x_1 + x_2 = 20,\ 2x_1 + 4x_2 = 50.$ Subtracting 2 times the first equation from the second one, we see that $2x_2 = 10.$ So, $x_2 = 5.$ By equation 1, $x_1 = 15.$

More Questions (Consistent or inconsistent?)

- Find the intersection of the lines $x_1 + x_2 = 20$ and $2x_1 + 4x_2 = 50$ on the (x_1, x_2)-plane or $\mathbb{R}^2$?
- What about the intersection of $x_1 + x_2 = 20$ and $2x_1 + 2x_2 = 50$?
- What about the intersection of $x_1 + x_2 = 20$ and $2x_1 + 2x_2 = 40$?

Matrix notation

Consider the following linear system and the corresponding augmented matrix:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

Taking away the last column, we have the coefficient matrix of the system.

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}$$

Row reduction of a matrix

Elementary row operations

1. Adding/subtracting a multiple of a row from another one.
2. Interchange two rows.
3. Multiply a row by a nonzero number.

Remark One may also reduce the augmented matrix to *reduced row echelon form* so that the leading one in each row is the only nonzero entry in its column.

Study Examples 1, 2, 3. (p.5–8.) Practice problems # 1 – 4. (p.9.)

No solution, one solution, infinitely many solutions

- In the row echelon form, we can identity: leading ones, pivot entries/positions, pivot columns, basic variables.
- The rest are non-pivot columns, free variables.
- Example: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 7 \end{bmatrix}$ Leading ones at $(1,1),(2,3)$ positions. Pivot columns: 1, 3. Basic variables: x_1, x_3.
- Three possibilities in the row echelon form.
  1. Inconsistent, i.e., no solution - there is a row with the last entry as a leading one.
  2. Consistent. No row with the last entry as a leading one.
     - 2a. One solution - No free variables.
     - 2b. Infinitely many solutions - There are free variables.
- In 2), we can find solutions using back substitution and write down the solutions in parametric form in 2b.

Study Examples 4, 5. (p. 18–20.) Practice Problems 1 and 2. (p.21.)

Reduced Row Echelon Form

- One can also use elementary operators to reduce an augmented matrix to reduced row echelon form so that the leading ones are the only nonzero entries in their columns.
- Example: $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 7 \end{bmatrix}$. So, $x_2 = 7, x_1 = -6$.
- Example: $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -6 \\ 0 & 0 & 1 & 7 \end{bmatrix}$.

  So, $x_3 = 7, x_2 = t$ with $t \in \mathbb{R}$, and $x_1 = -6 - t$.
- In practice, back substitutions is more efficient.
- Echelon forms of an augmented matrices may be different:

  Example: $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1/2 & 7 \end{bmatrix}$. So, $x_2 = 14, x_1 = -14/2 = -7$.

  $\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 7 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 7 \\ 0 & 1 & 14 \end{bmatrix}$; $\quad x_2 = 14, x_1 = 7 - (14) = -7$.
- Reduced row echelon form is unique. (Proof? Extra Credits!)

Summary

- Suppose a linear system is given.
- Write down the augmented matrix.
- Apply row reductions to get the (reduced) row echelon form.
- Three cases:

(1) (Inconsistent) There is a row $[0, \cdots, 0, b]$ for a nonzero b. Then the system has no solution.

(2) (Consistent) Condition (1) does not happen.

(2.a) Every variable is a leading/pivoting variable.

Then there is a unique solution determined by back substitution.

(2.b) There are non-leading / non-pivoting variables (columns).

There will be infinitely many solutions.

One can express the solutions in parametric form in terms the free variables by back substitutions.