Linear Algebra: Section 1.3 & 1.5

Objective:

Build connections between vector equations in $\mathbb{R}^n$ and linear systems.

Set notation Let S be a set, a collection of (mathematical) objects.
We write $a \in S$ if a is an element of S;
$T \subseteq S$ if every element of T is an element of S.

Example $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 7 \end{bmatrix}$. So, $x_2 = 7, x_1 = -6$.

The solution set of the system is: $\left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} : x_1 = -6, x_2 = 7 \right\} = \left\{ \begin{bmatrix} -6 \\ 7 \end{bmatrix} \right\}$.

Example $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & -6 \\ 0 & 0 & 1 & 7 \end{bmatrix}$.

So, $x_3 = 7, x_2 = t$ with $t \in \mathbb{R}$, and $x_1 = -6 - t$.

The solution set of the system is:

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = -6 - t, x_2 = t, x_3 = 7, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} -6 - t \\ t \\ 7 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

- Vectors in $\mathbb{R}^m$ for $m = 2, 3, \ldots$.
- We can define the sum of two vectors $\mathbf{u} + \mathbf{v}$, and the scalar multiple $c\mathbf{u}$, where $c \in \mathbb{R}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^m$.
- Linear combinations of two vectors in $\mathbb{R}^2$.
- Algebraic properties. p 27.
  (i) - (iv) about $\mathbf{u} + \mathbf{v}$; (v) – (viii) about scalar product $c\mathbf{u}$.
- The vector $\mathbf{y}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_n \in \mathbb{R}^m$ with weights $c_1, \ldots, c_n \in \mathbb{R}$ if
  $$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n.$$
- Study Examples 1 – 7 (p.25 – 31). Practice problems 1-2 (p.31).

Question Given $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b} \in \mathbb{R}^m$, can we express $\mathbf{b}$ as a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$? How do one find the weights $c_1, \ldots, c_n$?

Theorem 3 The vector $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$ if and only if the matrix equation / linear system $[\mathbf{a}_1 \cdots \mathbf{a}_n]\mathbf{x} = \mathbf{b}$ is solvable. The entries of a solution vector $\mathbf{x}$ gives the weights.

Proof. ... □

Remark We say that a set of vectors $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ spans/generates $\mathbb{R}^m$ if every vector in $\mathbb{R}^m$ is a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$.
In such a case, we write:

$$\mathbb{R}^m = \text{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

More results

Theorem 4 TFAE for an $m \times n$ matrix A.

(a) $Ax = \mathbf{b}$ always has a solution.

(b) Every $\mathbf{b}$ is a combination of the columns of A.

(c) The columns of A span $\mathbb{R}^m$.

(d) A has a pivot position in every row.

Proof. □

Theorem 5 Suppose A is $m \times n$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $c \in \mathbb{R}$.

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;

(b) $A(c\mathbf{u}) = c(A\mathbf{u})$.

Proof. □

Practice problems 1, 2.

Homogeneous linear system $A\mathbf{x} = \mathbf{0}$

- It always has a trivial solution $\mathbf{x} = \mathbf{0}$.
- The solutions can be written in parametric vector form.
- Nonhomogeneous linear systems, and solution sets.
- Let $\mathbf{p}, \mathbf{v} \in \mathbb{R}^m$. Vectors of the form $t\mathbf{v}$ with $t \in \mathbb{R}$ forms a line passing through $\mathbf{0}$; vectors of the form $\mathbf{p} + t\mathbf{v}$ with $t \in \mathbb{R}$ forms a line passing through $\mathbf{p}$ parallel to $\mathbf{v}$.
- Let $\mathbf{p}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ such that $\mathbf{u}$ and $\mathbf{v}$ are not multiple of each other. Then vectors of the form $\mathbf{p} + t_1\mathbf{u} + t_2\mathbf{v}$ with $t_1, t_2 \in \mathbb{R}$ correspond to a plane in $\mathbb{R}^m$.

Theorem 6 Suppose $A\mathbf{x} = \mathbf{b}$ has a solution p. Then all solutions have the form $\mathbf{p} + \mathbf{v}$, where $\mathbf{v}$ is a solution of the system $A\mathbf{x} = \mathbf{0}$.

Proof. ... □

Summary

Relation between linear equation and vector equation:

Let A be $m \times n$ with columns $\mathbf{v}_1, \dots, \mathbf{v}_n$, $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$.

Then $A\mathbf{x} = \mathbf{b}$ is the same as $x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_n = \mathbf{b}$.

The vector $\mathbf{b}$ is in Span$\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ if $A\mathbf{x} = \mathbf{b}$ is consistent.

The Span$\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is $\mathbb{R}^m$ if and only if A has m pivoting 1.

The solution set of $Ax = \mathbf{b}$ can be written in vector form:

$$\mathbf{v}_0 + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k \text{ with } t_1, \dots, t_k \in \mathbb{R}.$$

Here $\mathbf{v}_0$ is a particular solution for $A\mathbf{v}_0 = \mathbf{b}$,

and $t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$ are the general solution for $A\mathbf{x} = \mathbf{0}$.

If $k = 1$, we get a line; if $k = 2$, we get a plane in $\mathbb{R}^n$.