Linear Algebra: Section 1.7 & 1.9

Objective:

Use the theory of linear equations to study vectors & linear transformations

Linearly independent / dependent sets

- A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is linearly dependent if there are $x_1, \ldots, x_p \in \mathbb{R}$ not all zero such that

$$\mathbf{0} = x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = [\mathbf{v}_1 \cdots \mathbf{v}_p] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

  That is the homogeneous system has non-trivial solution.

  Example:

- Otherwise, the set is linearly independent, i.e., the homogeneous system only has zero solution.

  Example:

- One may consider the columns of a matrix A and discuss their linearly independence.

  Example:

Results for checking linearly independence

Observations (Easy cases)

- A set $\{\mathbf{v}\} \subseteq \mathbb{R}^n$ is linearly independent if and only if $\mathbf{v} \neq 0$.
- A set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly dependent if and only if $\mathbf{v}_2 = a\mathbf{v}_1$ or $\mathbf{v}_1 = b\mathbf{v}_2$ for some $a, b \in \mathbb{R}$.

Theorem 7 A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ with $p > 1$ is linearly dependent if and only if one of the vectors is a linear combination of the other vectors.

Proof. If $\mathbf{v}_i = c_1\mathbf{v}_1 + \cdots + c_{i-1}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_{i+1} + \cdots c_n\mathbf{v}_n$,
then setting $c_i = -1$ we have $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$.

If $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$ such that not all $c_1, \ldots, c_n$ are zero, say, $c_i \neq 0$, then

$$-c_i\mathbf{v}_i = c_1\mathbf{v}_1 + \cdots + c_{i-1}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_{i+1} + \cdots + c_n\mathbf{v}_n.$$

So,

$$\mathbf{v}_i = \frac{-c_1}{c_i}\mathbf{v}_1 + \cdots \frac{-c_{i-1}}{c_1}\mathbf{v}_{i-1} + \frac{-c_{i+1}}{c_1}\mathbf{v}_{i+1} + \cdots + \frac{-c_n}{c_1}\mathbf{v}_n.$$

□

Example

Theorem 8 Let $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} \subseteq \mathbb{R}^n$. If $p > n$, then the set is linearly dependent.

Proof. Think about 4 vectors in $\mathbb{R}^3$. Then consider solving $[\mathbf{v}_1 | \cdots | \mathbf{v}_p | \mathbf{0}]$. □

Theorem 9 If $S \subseteq \mathbb{R}^n$ contains the zero vector, then S is linearly dependent.

Proof. If $\{\mathbf{v}_1, \dots, \mathbf{v}_p\} \subseteq \mathbb{R}^n$ with $\mathbf{v}_i = \mathbf{0}$, then .... □

Practice problem 1 – 4 (p. 60).

Matrix transformations

Let A be an $m \times n$ matrix. Then we can define a transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ by $x \mapsto Ax$ such that for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n, c \in \mathbb{R}$,

(a) $A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v}$, and (b) $A(c\mathbf{u}) = cA\mathbf{u}$.

Examples

Geometrical meaning

Shear Transformation on $\mathbb{R}^2$

Linear transformations

Definition A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if for any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n, c \in \mathbb{R}$,

(a) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and (b) $T(c\mathbf{u}) = cT(\mathbf{u})$.

Remark If $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, then

(a) $T(\mathbf{0}) = \mathbf{0}$, and

(b) for any $\mathbf{v}_1, \ldots, \mathbf{v}_p \in \mathbb{R}^n, c_1, \ldots, c_p \in \mathbb{R}$,

$$T(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \cdots + c_pT(\mathbf{v}_p).$$

Turning the idea around, if we know $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$, then we know

$$T(\mathbf{b}) = b_1T(\mathbf{e}_1) + \cdots + b_nT(\mathbf{e}_n).$$

Results on linear transformations

Theorem 10 Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis for $\mathbb{R}^n$. Then

$$A = [T(\mathbf{e}_1) \cdots T(\mathbf{e}_n)]$$

is the unique $m \times n$ matrix satisfying

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \in \mathbb{R}^n.$$

Proof.

$$T(x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + \cdots + x_nT(\mathbf{e}_n) = [T(e_1) \cdots T(e_n)]\mathbf{x}.$$

Examples $\mathbb{R}^2$ - reflections, rotations, etc.

Surjective linear transformations/maps

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **onto/surjective** if for every $\mathbf{b} \in \mathbb{R}^m$ there is $\mathbf{x} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{b}$,

i.e., $A\mathbf{x} = \mathbf{b}$ is solvable for all $\mathbf{b} \in \mathbb{R}^m$.

Theorem 11

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is surjective if and only if its standard matrix A of transformation span $\mathbb{R}^m$.

Examples:

Injective linear transformations

A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **one-one/injective** if $T(\mathbf{u}) \neq T(\mathbf{v})$ whenever $\mathbf{u} \neq \mathbf{v}$ in $\mathbb{R}^n$,

i.e., $T(\mathbf{u} - \mathbf{v}) \neq \mathbf{0}$ in $\mathbb{R}^m$ whenever $\mathbf{u} - \mathbf{v} \neq \mathbf{0}$ in $\mathbb{R}^n$,

i.e., $T(\mathbf{x}) \neq \mathbf{0}$ in $\mathbb{R}^m$ whenever $\mathbf{x} = \mathbf{0}$ in $\mathbb{R}^n$,

i.e., $T(\mathbf{x}) = \mathbf{0}$ implies that $\mathbf{x} = \mathbf{0}$.

Theorem 12

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ with standard matrix A. The following are equivalent.

(1) T is one-one,

(2) $T(\mathbf{x}) = \mathbf{0}$ only has trivial solution.

(3) the columns of A are linearly independent.

Practice problems 1 – 3. (p.68)

Practice problem. p.77.