Linear Algebra: Section 2.1 - 2.2

Objective:

Study basic matrix operations and invertible matrices.

Notation

- An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns, often denoted by $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$.
- One can discuss the (i, j), the diagonal entries, etc.
- We often use diagonal matrices, identity matrix, zero matrix, etc.

Theorem 1 Let A, B, C be matrices of the same size, r, s be scalars. Then

(a) $A + B = B + A$, (b) $(A + B) + C = A + (B + C)$,

(c) $A + O = O + A$, (d) $r(A + B) = rA + rB$,

(e) $(r + s)A = rA + sA$, (f) $(rs)A = r(sA)$.

Proof. Note that the (i, j) entry of $X + Y = x_{ij} + y_{ij}$, and

$X = Y$ if they have the same (i, j) entries for each (i, j) pair.

Therefore, ...

Recall: If A has columns $v_1, \ldots, v_n$ and $x \in \mathbb{R}^n$ has entries $x_1, \ldots, x_n$, then $Ax = x_1 v_1 + \cdots + x_n v_n$.

Definition (Matrix multiplication)

Let A be an $m \times n$ matrix, and $B = [b_1|\cdots|b_p]$ be an $n \times p$ matrix. Then

$$AB = [Ab_1|\cdots|Ab_p]$$

is an $m \times p$ matrix.

Examples ...

Remark If $A = \begin{bmatrix} \mathrm{row}_1(A) \\ \vdots \\ \mathrm{row}_n(A) \end{bmatrix}$ then $AB = \begin{bmatrix} \mathrm{row}_1(A)B \\ \vdots \\ \mathrm{row}_n(A)B \end{bmatrix}$.

Proof. Check the (i, j) entry!

Theorem 1

Let A, B, C be matrices of suitable sizes, r be a scalar. Then

(a) $(AB)C = A(BC)$ (b) $A(B + C) = AB + AC$,

(c) $(B + C)A = BA + CA$, (d) $r(AB) = (rA)B$,

(e) $I_m A = A = AI_n$ if A is $m \times n$.

Proof. (a) Let $A = [A_1 \cdots A_n]$ be $m \times n$, $B = [B_1 \cdots B_p]$ be $n \times p$, and

$$C = [C_1 \cdots C_q] = [c_{ij}] \text{ be } p \times q. \text{ Then } AB = [Ab_1| \cdots |Ab_p].$$

We show that the kth column of $(AB)C$ equals to that of $A(BC)$...

(b) We show that the kth column of $A(B + C)$ equals that of $AB + AC$...

Remarks and Examples

(1) $AB \neq BA$ in general.

(2) $AB = AC$ does not imply that $B = C$.

(3) $AB = 0$ does not imply $A = 0$ or $B = 0$.

More terminology/operations

If $v \in \mathbb{R}^n$ has entries $v_1, \ldots, v_n$, then the transpose of v is $v^T = [v_1 \cdots v_n]$.

Let $A = [A_1 \cdots A_n]$ be $m \times n$. Then the transpose of A is A^T is $n \times m$ with rows $A_1^T, \ldots, A_n^T$.

If A is $n \times n$, then $A^r = \underbrace{A \cdots A}_{r}$ for any positive integer r.

Theorem 3 Let A, B be matrices of appropriate sizes, r be a scalar.

(a) $(A^T)^T = A$.

(b) $(A + B)^T = A^T + B^T$,

(c) $(rA)^T = rA^T$,

(d) $(AB)^T = B^T A^T$.

Proof of (d). The (i, j) entry of $B^T A^T$ equals $\mathrm{Row}_i(B)\mathrm{Col}_j(A)$.

The (i, j) entry of $(AB)^T$ is the (j, i) entry of AB and equals

$$\mathrm{Row}_j(A)\mathrm{Col}_i(B).$$

Invertible matrices

Definition A square matrix A is invertible if there is a matrix B such that

$$AB = BA = I.$$

If such a B exists, it is unique, and is called the inverse of A, denoted by A^{-1}.

Theorem 4 Let $A = [a_{ij}]$ be 2×2. If $\det(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0$, then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix}.$$

Proof. Direct checking.

Theorem 5 If A is an invertible $n \times n$ matrix, then for each $b \in \mathbb{R}^n$ the linear system $Ax = b$ has a unique solution $x = A^{-1}b$.

Proof. If $Ax = b$, then $x = (A^{-1}A)x = A^{-1}(Ax) = A^{-1}b$.
If $x = A^{-1}b$, then $Ax = A(A^{-1}b) = (AA^{-1})b = b$.

Efficient algorithm for computing inverse

Theorem 7 Row reductions on $[A|I_n]$ yield $[I_n|A^{-1}]$ if A is invertible.

Proof. Let $A^{-1} = B = [B_1 \cdots B_n]$.

Then $AB_k = e_k$ for $k = 1, \ldots, n$,

and B_k is obtained by row reduction of $[A|e_k]$ to $[I_n|B_k]$.

Thus, we may row reduce $[A|I_n]$ to $[I_n|B_1 \cdots B_n] = [I_n|B] = [I_n|A^{-1}]$.

Example ....

Theorem 6 Let A, B be invertible $n \times n$ matrices.

(a) $(A^{-1})^{-1} = A$.

(b) $(AB)^{-1} = B^{-1}A^{-1}$ (mind the order).

(c) $(A^T)^{-1} = (A^{-1})^T$.

Proof. (a) Let $C = (A^{-1})^{-1}$ so that $I = A^{-1}C$.

Multiplying both sides by A, we get $A = C$.

(b) Let $C = (AB)^{-1}$ so that $I = (AB)C$.

Multiplying both sides by $B^{-1}A^{-1}$, we see that $B^{-1}A^{-1} = C$.

(c) Let $C = (A^T)^{-1}$ so that $I = A^TC$. Multiplying both sides by $(A^{-1})^T$,

we see that $(A^{-1})^T = (A^{-1})^TA^TC = (AA^{-1})^TC = C$.

Elementary matrices

There are three types of elementary matrices obtained from I_n.

1) Interchanging the ith and jth row (of the identity matrix).
2) Adding $r \neq 0$ times the ith row to the jth row (of the identity matrix).
3) Multiplying the ith row by $r \neq 0$.

Remark

If E is an elementary matrix, then EA corresponds to the elementary row operation to A. We can use this to prove that $AB = I$ ensures $BA = I$.

Theorem If there are elementary operations $E_1, \ldots, E_k$ such that

$$E_k \cdots E_1 [A|I] = [I_n | E_k \cdots E_1] = [I|B],$$

then $BA = I = AB$.

Proof. Applying the row reduction to A to get I_n is the same as multiplying A with the corresponding elementary matrices. So, $BA = E_k \cdots E_1 A = I_n$. Note that $AB_i = e_i$ for each $i = 1, \ldots, n$. Thus $AB = I_n$. □