Linear Algebra: Section 2.3 - 2.5

Objective: Study invertible matrices, block matrices, matrix factorization

Conditions for invertibility

Theorem 8 The following are conditions are equivalent for an $n \times n$ matrix A.

a. A is invertible.

b. A is row equivalent to the I_n.

c. A has n pivot positions.

d. The equation $Ax = 0$ has only trivial solution.

e. The columns of A are linearly independent.

f. The linear transformation $x \mapsto Ax$ is one-to-one, i.e., $x \neq y$ ensures $Ax \neq Ay$.

g. The equation $Ax = b$ has at least one solution.

h. The columns of A span $\mathbb{R}^n$.

i. The linear transformation $x \mapsto Ax$ is onto.

j. There is C such that $CA = I$.

k. There is D such that $AD = I$.

l. A^T is invertible.

Inverse transform

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation. Then T is invertible if there is a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S \circ T(u) = u$ and $T \circ S(v) = v$ for any $u, v \in \mathbb{R}^n$. That is, for $u, v \in \mathbb{R}^n$, $S(v) = u$ if and only if $T(u) = v$.

Theorem 9 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ with A as the standard matrix. Then T is invertible if and only if A is invertible. In such a case, $S(x) = A^{-1}x$ is the inverse transformation of T.

Proof. Suppose T is invertible. Let A be the matrix of T and B be the matrix of S. Then $S \circ T(x) = B(Au) = u$ and $T \circ S(v) = A(Bv)$.

Block matrices

We can partition a matrix to get a block matrix, and perform multiplication on block matrices of appropriate sizes.

Applications

- If $AB = \mathrm{col}_1(A)\mathrm{row}_1(B) + \cdots + \mathrm{col}_n(A)\mathrm{row}_n(B)$.
- If $A = \begin{bmatrix} I_p & A_{12} \\ O & I_q \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} I_p & -A_{12} \\ O & I_q \end{bmatrix}$
- If $A = \begin{bmatrix} A_{11} & A_{12} \\ O & A_{22} \end{bmatrix}$ such that A_{11} and A_{22} are invertible, then $A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ O & A_{22}^{-1} \end{bmatrix}$.

Let A be $n \times n$. Then one can apply special elementary row operations to A to get row echelon form: $(E_p \cdots E_1)A = U$, where $E_p \cdots E_1$ is in lower triangular form with all diagonal entries equal to 1. Then

$$A = (E_p \cdots E_1)^{-1} U = LU.$$

Application To solve $Ax = b$, we consider $LUx = b$, and solve $Ly = b$ and then $Ux = y$. Each system can be solved by back substitution.

Summary of Chapters 1 and 2

- Linear equation. Augmented matrix. No solution. One solution. Infinitely many solutions.
- Pivoting positions in rows, in columns.
- Vector equations. A linear combination $v = c_1 v_1 + \cdots + c_n v_n \in \mathbb{R}^m$. Solve a corresponding linear system: $Ax = v$ with $A = [v_1 \cdots v_n]$.
- A set of vectors $\{v_1, \ldots, v_p\}$ in $\mathbb{R}^m$ is linear dependent if there are non-trivial solution for $0 = c_1 v_1 + \cdots + c_p v_p$, i.e., $Ax = 0$ has non-trivial solution for $A = [v_1 \cdots v_p]$. Else, the set is linearly independent, i.e., the linear system has non-trivial solutions.
- Matrix operations: $A + B, AB, rA, A^T$.
- Inverse of a square matrix A; and many equivalent conditions.
- Block matrices and factorization. $A = LU$.