Linear Algebra: Section 3.1–3.3

Objective: Study determinant - a useful concept for previous and future study.

Working definition of determinant

A definition allows efficient computation of determinant.

Let $A = (a_{ij})$ be $n \times n$.

- If A is upper triangular, then $\det(A) = a_{11} \cdots a_{nn}$.
- For a general $A = (a_{ij})$, apply row operations of

  type I (interchange rows) and

  type II (subtract a multiple of the ith row from the jth row)

  to get an upper triangular matrix T.
  Then
  $$E_k \cdots E_1 [A | I_n] = [T | E_k \cdots E_1]$$
  and $\det(A) = (-1)^r \det(T)$ where r is the number of type I matrices used in the reduction.
- Thus, $\det(A) = 0$ if A has fewer than n pivoting positions, $\det(A) = (-1)^r$(product of pivots).

A recursive definition and cofactor expansion

Examples 2×2, 3×3.

Definition If $n = 1$, $\det([a]) = a$. For $n \geq 2$,

$$\det(A) = \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det A_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij},$$

where A_{ij} is obtained from A by removing the ith row and the jth column, and $C_{ij} = (-1)^{i+j} \det(A_{ij})$ is the (i, j)-cofactor of A.

Rework previous examples

Elementary matrices and determinant

Let E be an $n \times n$ elementary matrix, and A, B be $n \times n$ matrices.

- If E is the obtained from I_n by interchanging the ith and jth row, then $\det(E) = -1$.
- If E is obtained from I_n by adding a multiple of the ith row to another row, then $\det(E) = 1$.
- If E is obtained from I_n by multiplying the ith row by $r \neq 0$, then $\det(E) = r$.
- We have $\det(EA) = \det(E)\det(A)$.
- So, $\det(tA) = t^n \det(A)$.

Product of matrices

Theorem Let A, B be $n \times n$ matrices. Then $\det(AB) = \det(A)\det(B)$.

Proof. Case 1. A is singular and $\det(A) = 0$. Then are elementary matrices $E_1, \dots, E_k$ such that $E_k \cdots E_1 A$ has a zero row, and so is $E_k \cdots E_1 AB$. So, $\det(AB) = 0$.

Case 2. A is invertible. Then there are elementary matrices $E_1, \dots, E_k$ such that $E_k \cdots E_1 A = I_n$. Thus, $1 = \det(I_n) = \det(E_k) \cdots \det(E_1)\det(A)$ so that

$$1/(\det(A)) = \det(E_k) \cdots \det(E_1) = \det(E_k \cdots E_1) = \det(A^{-1}).$$

It follows that

$$\det(B) = \det(I_n B) = \det(A^{-1}AB)$$
$$= \det(E_k \cdots E_1 AB) = \det(A^{-1})\det(AB) = \det(AB)/\det(A).$$

Additional Properties

Let A, B be $n \times n$.

- We have $\det(A) = \det(A^T)$ so that $\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij}$.
  Proof. If $A = E_1 \cdots E_k U$,
- If A is invertible, then $\det(A)\det(A^{-1}) = 1$.
- (Linearity in each row or column) For any $i \in \{1, \ldots, n\}$, if $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have

$$\det(A_i(u+v)) = \det(A_i(u)+A_i(v)) \quad \text{and} \quad \det(A_i(cu)) = c\det(A_i(u)).$$

Remark In general, $\det(A+B) \neq \det(A) + \det(B)$.

Other applications

- (Cramer's rule) If A is invertible and if $Ax = b$, then $x_i = \det(A_i(b))/\det(A)$, where $A_i(b)$ is obtained from A by replacing its ith column with b.

  Proof. Note that $x = A^{-1}b$, and $x_i = \det(A)^{-1}\det(A_i(b))$. (**Why?**)

- (Adjoint formula for inverse) If A is invertible, then $A^{-1} = (\det A)^{-1}(C_{ij})^T$, where C_{ij} is the (i,j) cofactor of A.

  Proof. Note that $A(C_{ij}) = \det(A)I_n$.

- The area (volume) of parallelogram (parallelepiped) P determined by the 2×2 (3×3) matrix B equals $|\det(B)|$.

  Examples

- Moreover, under the transformation $x \mapsto Ax$, then area (volume) will change from $\det(B)$ to $\det(A)\det(B)$.

  Examples