Linear Algebra: Section 4.1–4.3

Objective: Introduction of vector spaces, subspaces, and bases.

Vector space

Vector space Examples: $\mathbb{R}^n$, subsets of $\mathbb{R}^n$, the set of polynomials (up to degree n), the set of (continuous, differentiable) real valued functions, etc.

Definition A vector space is a (non-empty) set with elements called vectors equipped with an addition of vectors $\mathbf{u}+\mathbf{v}$ and a scalar multiplication of vectors by (real) scalars $c\mathbf{u}$ such that for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ and scalars c and d:

1. $\mathbf{u}+\mathbf{v} \in V$. 2. $\mathbf{u}+\mathbf{v} = \mathbf{v}+\mathbf{u}$. 3. $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$.

4. There is a zero vector $\mathbf{0}$ such that $\mathbf{0}+\mathbf{v}=\mathbf{v}$.

5. For every $\mathbf{v}$ there is $-\mathbf{v} \in V$ such that $\mathbf{v}+(-\mathbf{v}) = \mathbf{0}$.

6. $c\mathbf{v} \in V$. 7. $c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v}$. 8. $(c+d)\mathbf{v} = c\mathbf{v}+d\mathbf{v}$.

9. $c(d\mathbf{v}) = (cd)\mathbf{v}$. 10. The number 1 satifies $1\mathbf{v} = \mathbf{v}$.

Properties. For any $\mathbf{u} \in V$ and scalar c:

$$0\mathbf{u} = \mathbf{0}, \qquad c\,\mathbf{0} = \mathbf{0}, \qquad -\mathbf{u} = (-1)\mathbf{u}.$$

Examples

$\mathbb{R}^n$,

$\mathbb{R}^{m\times n}$,

$\mathbb{R}^{1\times n}$,

$\mathbb{P}_n(t)$,

$\mathbb{P}(t)$,

Set of continuous function on $\mathbb{R}$ or on an interval,

Set of differentiable function on $\mathbb{R}$ or on an interval.

Let A be $m \times n$. The solution set of $Ax = 0$ in $\mathbb{R}^n$.

Let A be $m \times n$. The set of linear combinations of the columns of A.

Subspaces

Definition A subset H of a vector space V is a subspace if H is a vector space under the same operations.

Remark One needs only check three things:

(a) $\mathbf{0} \in H$, (b) $\mathbf{u} + \mathbf{v} \in H$, and (c) $c\mathbf{u} \in H$ for any scalar c, and $\mathbf{u}, \mathbf{v} \in H$.

Subspaces related to a matrix

Let A be $m \times n$. The null space $\operatorname{Nul} A$ and the column space of A are:

$$\operatorname{Nul} A = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = 0\} \subseteq \mathbb{R}^n \quad \text{and}$$

$$\operatorname{Col} A = \operatorname{span}\{a_1, \ldots, a_n\} = \{A\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m.$$

Kernel and Range of a Linear Transformation

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T(\mathbf{x}) = A\mathbf{x}$. Then the kernel of T is

$$\operatorname{kernel}(T) = \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

The range space of T is

$$\operatorname{range}(T) = \{T(\mathbf{x}) \in \mathbb{R}^m : \mathbf{x} \in \mathbb{R}^n\} = \{A\mathbf{x} \in \mathbb{R}^m : \mathbf{x} \in \mathbb{R}^n\}.$$

Linearly independent sets

Definition A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subseteq V$ is linearly independent if the vector equation

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

only has trivial solution $c_1 = \cdots = c_p = 0$.

Theorem 4 Let $p \geq 2$. Then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subseteq V$ of vectors is linearly independent if and only if $\mathbf{v}_j$ is not a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$, for every $j > 1$.

Proof. If $\mathbf{v}_j = a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p$ without the $\mathbf{v}_j$ term on the right, then $0 = a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p$ where $a_j = -1$. So, $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent.

If $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly dependent, then $0 = a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p$ with at least one j such that $a_j \neq 0$. Then $-a_j\mathbf{v}_j = a_1\mathbf{v}_1 + \cdots + a_p\mathbf{v}_p$ without the $\mathbf{v}_j$ term on the right. Dividing both side by the scalar $-a_j$, we see that $\mathbf{v}_j$ is a linear combination of the other vectors. □

Bases

Definition Let H be a subspace of V. A basis for H is a linearly independent subset of H that spans H.

Theorem 6 The pivot columns of an $m \times n$ matrix A form a basis for $\mathrm{Col}\, A$.

Proof. Let $\tilde{A}$ be the submatrix of A using the pivot columns of A. Then $Ax = b$ has solution precisely when $\tilde{A}x = b$ has solution. □

Theorem 5 Suppose $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subseteq V$ that spans a subspace H.

(a) If $\mathbf{v}_k$ is a linearly combination of the other vectors in S, then H is also spanned by $S - \{\mathbf{v}_k\}$.

(b) If $H \neq \{\mathbf{0}\}$, then a subset of S forms a basis for H.

Proof. (a) If v is a linear combination of S, then it is a linear combination of vectors in $S - \{v_k\}$.

(b) The smallest spanning set chosen from S will be linearly independent and spanning. □

Remark Let $S \subseteq V$. Then S is a basis if any one of the following is true.

(a) S is a maximal linearly independent set.

(b) S is minimal spanning set.