Linear Algebra: Section 4.4–4.7

Objective: Study Coordinate systems, rank, dimension, change of bases.

Key idea Assume we know $Ax = b$, Nul$A$, Col$A$, $T(x) = Ax$ well. Convert problems of vector spaces to matrix computation using basis, and choose a "good" basis.

Coordinate systems

Remark First we learn how to choose different basis in $\mathbb{R}^n$.

Theorem 7 Let $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be a basis for V. Then every $\mathbf{x} \in V$ has a unique representation $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Definition The values $c_1, \ldots, c_n$ are the $\mathcal{B}$-coordinates of $\mathbf{x}$, and we write

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Coordinates in $\mathbb{R}^n$

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, and $\mathbf{x} \in V$. If $P_{\mathcal{B}} = [\mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_n]$, then $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$.

Theorem 8 The mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is an invertible linear transformation.

Remark The matrix $P_{\mathcal{B}}$ is the change-of-ordinate matrix from $\mathcal{B}$ to the standard basis in $\mathbb{R}^n$, and $P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$. In general, if $\mathcal{B}$ and $\mathcal{C}$ are bases for $\mathbb{R}^n$, then

$$x = P_{\mathcal{B}}[x]_{\mathcal{B}} = P_{\mathcal{C}}[x] \quad \text{so that} \quad [x]_{\mathcal{C}} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}[x]_{\mathcal{B}}.$$

The matrix

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1}P_{\mathcal{B}}$$

is the unique matrix such that

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}.$$

Note that

$$[\mathbf{b}_i]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{b}_i]_{\mathbf{B}} = P_{\mathcal{C}\leftarrow\mathcal{B}}e_i,$$

which is the ith column of $P_{\mathcal{C}\leftarrow\mathcal{B}}$. Thus,

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \left[[\mathbf{b}_1]_{\mathcal{C}}[\mathbf{b}_2]_{\mathcal{C}}\cdots[\mathbf{b}_n]_{\mathcal{C}}\right].$$

See Theorem 15.

Coordinate systems for general vector spaces

Theorems 9–10 Suppose V has a basis containing n vectors.

(a) Any set in V with more than n vectors is linearly dependent.
(b) Every basis of V has exactly n vectors.

In such a case, we say that V has dimension n.

Proof. Let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a basis for V. Suppose $S = \{v_1, \ldots, v_p\}$ with $p > n$.
Then $v_i = a_{i1}b_1 + \cdots + a_{in}b_n$ for $i = 1, \ldots, p$.
To solve $c_1, \ldots, c_p$ such that

$$0 = c_1v_1 + \cdots + c_pv_p = \sum c_i(a_{i1}b_1 + \cdots + a_{in}b_n)$$

$$= (a_{11}c_1 + \cdots + a_{p1}c_p)b_1 + \cdots + (a_{1n}c_1 + \cdots + a_{pn}c_p)b_n$$

so that

$$A\begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n,$$

where A is $n \times p$.
So, there is a nontrivial solution $(c_1, \ldots, c_n)$.
(b) Now, if $\mathcal{B}'$ is another basis, then $\mathcal{B}'$ has at most m vectors with $m \leq n$.
Interchanging the roles of $\mathcal{B}$ and $\mathcal{B}'$, we see that $n \leq m$. So, $m = n$. $\square$

Dimension of a vector space

Remark If $V = \{\mathbf{0}\}$, then V has 0 dimension; if V does not have a finite basis, it is infinite dimensional.

Theorem 11 If V is finite dimensional and H is a subspace of V, then $\dim H \leq \dim V$.

Proof. Suppose V has a basis $\{b_1, \ldots, b_n\}$. If H has a basis S, then S cannot have more than n vectors, else, it is linearly dependent. □

Remark Let A be $m \times n$. Then $\dim \mathrm{Col}\, A = \dim \mathrm{Row} A$ equals the number of pivoting positions, $\dim \mathrm{Nul} A$ equals the number of free variables.

The rank of A is the number of pivoting positions, i.e., $\dim \mathrm{Col}\, A = \dim \mathrm{Nul} A$.

As a result, $\mathrm{rank}\, A + \dim \mathrm{Nul} A = n$.

Invertibility

Theorem - Additional conditions for invertibility An $n \times n$ matrix is invertible if any one of the following holds.

(m) The columns of A form a basis for $\mathbb{R}^n$.

(n) $\mathrm{Col}\, A = \mathbb{R}^n$.

(o) $\dim \mathrm{Col}\, A = n$.

(p) rank $A = n$.

(q) $\mathrm{Nul} A = \{\mathbf{0}\}$

(r) $\dim \mathrm{Nul} A = 0$.