Linear Algebra: Sections 5.1 – 5.5

Objective: Change a matrix A to a simple form by similarity.

Key idea Find eigenvalues and eigenvectors to form a basis so that a given matrix A can be changed to a simple form.

Eigenvalues, eigenvectors, and Characteristic equation

Let A be $n \times n$. A **nonzero** vector is an eigenvector of A if there is a scalar λ such that $Ax = \lambda x$. The value λ is an eigenvalue, and x is an eigenvector corresponding to λ.

Examples Let $A =$ ....

Question How to find eigenvalues and eigenvectors?

Special case: If $\det(A) = 0$, then $Ax = 0x$ for some nonzero x.

General case: If $\det(A - \lambda I) = 0$, then $(A - \lambda I)x = 0$ for some nonzero x.

Answer to the question
Find λ such that $\det(A - \lambda I) = 0$. Then solve the eigenvectors x corresponding to $(A - \lambda I)x = 0$.

The equation $\det(A - \lambda I) = 0$ is called the characteristic equation (polynomial) of A.

Examples Let $A =$ .....

Remarks

- There may be no real eigenvalues, but there are always complex eigenvalues. In the latter case, we may do complex arithmetic.
- There may be more than one linearly independent eigenvectors corresponding to an eigenvalue λ. The maximum number is the dimension of Nul $(A - \lambda I)$.
- If $x_1, \ldots, x_p$ satisfy $Ax_i = \lambda_i x_i$, then

$$A[x_1 \cdots x_p] = [x_1 \cdots x_p]D \quad \text{with} \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_p \end{bmatrix}.$$

- If $p = n$ and $S = [x_1 \ldots x_n]$ has linearly independent columns, then

$$A = SDS^{-1} \quad \text{and} \quad S^{-1}AS = D.$$

  In such a case:

  (4.1) $A^k = S^{-1}D^kS$;

  (4.2) every $x \in \mathbb{R}^n$ can be written as $x = c_1x_1 + \cdots + c_nx_n$ so that

$$Ax = c_1\lambda_1x_1 + \cdots + c_n\lambda_nx_n,$$

  i.e, $[Ax]_\mathcal{B} = D[x]_\mathcal{B}$ with $\mathcal{B} = \{x_1, \ldots, x_n\}$.

- So, we would like to find n linearly independent eigenvectors for A.
- In some cases, we will not be able to find n linearly independent eigenvectors so that A is not diagonalizable (by similarity).
- (Theorem 5) Thus, A is diagonalizable if and only if A has n linearly independent eigenvectors.
- (Theorem 4) Eigenvectors corresponding to different eigenvalues are linearly independent.
- We will focus on Nul$(A - \lambda I)$ for each eigenvalue λ.

Similarity

Definition Two $n \times n$ matrices A and B are similar if there is an invertible S such that $B = SAS^{-1}$.

Remark If A and B are similar, then A and B have the same determinant and the same eigenvalues.

Definition Suppose $T : V_1 \to V_2$ is a linear transformation, and suppose $\mathcal{B}_1 = \{u_1, \dots, u_n\}$ and $\mathcal{B}_2 = \{v_1, \dots, v_m\}$. Let

$$a_1 = [T(u_1)]_{\mathcal{B}_2}, \dots, a_n = [T(u_n)]_{\mathcal{B}_2}.$$

Then $A = [a_1 \cdots a_n]$ is the matrix of T relative to the bases $\mathcal{B}_1, \mathcal{B}_2$, and for any $x \in v_1$ we have

$$[T(x)]_{\mathcal{B}_2} = A[x]_{\mathcal{B}_1}.$$

If $T : V \to V$ and $\mathcal{B}_2 = \mathcal{B}_1$, then $A = [T]_{\mathcal{B}_1}$ is the matrix of T relative to $\mathcal{B}_1$.

Theorem 8 Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be such that $T(x) = Ax$. Suppose P is such that $P^{-1}AP = D$ a diagonal matrix. If we use the columns of P as the basis $\mathcal{B}$ for $\mathbb{R}^n$, then D is the matrix of T relative to $\mathcal{B}$.
Proof. Note that $P^{-1}x = [x]_{\mathcal{B}}$....

Remark If A is the matrix of $T : V \to V$ relative to a basis $\mathcal{C}$. If there is P such that $P^{-1}AP = D$ is a diagonal matrix. Then there is a basis $\mathcal{B}$ such that the matrix of T relative to $\mathcal{B}$ is D. In fact, we have

$$P^{-1}[x]_{\mathcal{C}} = [x]_{\mathcal{B}}.$$

Matrices with complex eigenvalues: An example.
Let $A = ...$

Eigenvectors and linear transformations

Definition Suppose V_1, V_2 are vector spaces with bases $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{C} = \{c_1, \ldots, c_m\}$, respectively. Then

$$M = [[T(b_1)]_{\mathcal{C}} \cdots [T(b_n)]_{\mathcal{C}}]$$

is an $m \times n$ matrix, known as the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$, such that

$$[T(u)]_{\mathcal{C}} = M[u]_{\mathcal{B}}.$$

Example Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x) = Ax$ with $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \end{bmatrix}$.

Then A is the matrix for T relative to the standard bases.

Suppose $\mathcal{B}$ is the standard basis for $\mathbb{R}^3$, $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$.

Then $M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

Example Let $T : \mathbf{P}_2(t) \to \mathbb{R}^2$ defined by

$$T(a_0 + a_1 t + a_2 t^2) = \begin{bmatrix} a_0 + 2a_1 + 3a_2 \\ 4a_0 + 5a_1 + 9a_2 \end{bmatrix}.$$

Then A is the matrix for T with respect to the standard bases.
One can use the standard basis $\mathcal{B} = \{1, t, t^2\}$ and $\mathcal{C}$ above so that M is the matrix for T relative to the bases $\mathcal{B}$ and $\mathcal{C}$.

Theorem If $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $T(x) = Ax$. If $\mathcal{B} = \{b_1, \ldots, b_n\}$ is a basis for $\mathbb{R}^n$ and $P = [b_1 \cdots b_n]$, then $P^{-1}AP$ is the $\mathcal{B}$-matrix for T, i.e., the matrix for T relative to $\mathcal{B}$,
Proof. Let $\mathcal{C}$ be the standard basis. For any $x \in \mathbb{R}^n$, we have

$$P^{-1}AP[x]_{\mathcal{B}} = S^{-1}A[x]_{\mathcal{C}} = S^{-1}[T(x)]_{\mathcal{C}} = [T(x)]_{\mathcal{B}}.$$

Remark If $\mathcal{B}$ consists of eigenvectors of A, then $P^{-1}AP = D$ is in diagonal form. In particular,

$$Tb_i = \lambda_i b_i \text{ for } i = 1, \ldots, n.$$

Remark Note that the same conclusion holds for any V with dimension n.

Example Let $T : \mathbf{P}_1(t) \to \mathbf{P}_1(t)$ defined by $T(a_0 + a_1 t) = (a_0 + a_1) + (a_0 + a_1)t$. Then using the standard basis $\mathcal{C} = \{1, t\}$ the $\mathcal{C}$ matrix is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be a basis consisting of eigenvectors of A. Then

$$P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $T(1+t) = 2(1+t)$, and $T(1-t) = 0(1-t)$. That is, for $u = 1+t$ and $v = (1-t)$ in $\mathbf{P}_1(t)$, we have $T(u) = 2u$ and $T(v) = 0v$