Math 214 Final Examination Name: Sample Solution

(1) (20 points, 5 points each) Answer the following short questions.

(a) Let $P(x) : x \geq 0$; $Q(x) : x^2 \geq 1$. Determine with explanation the truth set

$$T = \{x \in \mathbb{R} : \text{“}P(x) \Rightarrow Q(x)\text{” is true}\}.$$

If $x<0$ then "$P(x) \Rightarrow Q(x)$" is true

If $x \geq 0$ then "$Q(x)$ is true provided $x \geq 1$.

So the truth set $T = (-\infty, 0) \cup [1, \infty) = \mathbb{R} - [0, 1)$.

(b) Determine the last digit of $n = 3^{2017}$.

~~So~~ In $\mathbb{Z}_{10}$, $[3^{2017}] = [3^2]^{1008}[3] = [-1]^{1008}[3] = [3]$ #

$\therefore$ The last digit of 3^{2017} is 3.

(c) Prove or disprove the following: There exists an integer x so that $x^2 \equiv 2 \pmod 3$.

In $\mathbb{Z}_3$, $x = [0], [1], [2]$.

So that $[x^2] = [0], [1], [4] = [1]$.

So there is NO integer x so that $x^2 \equiv 2 \pmod{3}$

(d) Consider the bijections $f_1 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{pmatrix}$. Compute $f_1 \circ f_2$, and f_1^{-1}.

$f_1 \circ f_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$ $f_1^{-1} = \begin{pmatrix} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

(2) (10 points) (a) Use the Euclidean Algorithm to find $d = \gcd(22, 121)$.

(b) Show that there are infinitely many pairs of integers (x, y) satisfying $d = 22x + 121y$.

(a) $121 = 5 \times 22 + 11$

$22 = 2 \times 11 + 0$

$\gcd(22, 121)$
$= \gcd(22, 11)$
$= 11$

And

$11 = 121 - 5 \times 22$

Note that

$11 = 22x + 121y$
for $(x, y) = (-5, 1)$.

For every $n \in \mathbb{Z}$.
let $(x_n, y_n) = (-5-121n, 1+22n)$

Then $22(-5-121n) + 121(1+22n)$
$= 22(-5) + 121(1)$
$= 11$.

So there are infinitely many (x_n, y_n) s.t. $d = 22x_n + 121y_n$.

(3) (10 points) Prove that if a is an odd integer and $n = (a+4)^2 + (a+2)^2 + a^2 + 1$, then $12|n$.

[Hint: Show that both 3 and 4 are factors of $(a+4)^2 + (a+2)^2 + a^2 + 1$.]

Let $a = 2k+1$.

$n = (2k+1+4)^2 + (2k+1+2)^2 + (2k+1)^2 + 1$.

$= 4k^2 + 4 \cdot k \cdot 5 + 25 + 4k^2 + 4k \cdot 3 + 9 + 4k^2 + 4k + 1 + 1$

$= 4(k^2 + 5k + k^2 + 3k + k^2 + k) + 36$.

$= 4(3k^2 + 9k) + 36$.

Clearly $4 | n$ & $3 | n$.

So $12 \cancel{24} | n$.

(4) (10 points) Prove that $3^{2n} - 2^n$ is divisible by 7 for any nonnegative integer n.

Let $P(n)$: $7 \mid (3^{2n} - 2^n)$.

$n=0$, $7 \mid (1-1)$. $\therefore$ $P(0)$ holds.

Assume $P(k)$ holds, i.e., $3^{2k} - 2^k = 7m$ for some $m \in \mathbb{Z}$ with $k \geq 0$

Then $3^{2(k+1)} - 2^{(k+1)} = 3^{2k} \cdot 9 - 2^k \cdot 2$

$= 9(3^{2k} - 2^k) + 9 \cdot 2^k - 2 \cdot 2^k$

$= 9 \cdot 7m + 7 \cdot 2^k = 7(9m + 2^k)$ by induction assumption. which is divisible by 7

$\therefore$ $P(k+1)$ holds.

By principle of M. I., $P(n)$ holds for ~~$n \geq 0$~~, $n \in \mathbb{Z}, n \geq 0$.

(5) (10 points) Let $n \in \mathbb{N}$. Show that $\sqrt{n} \in \mathbb{Q}$ if and only if $n = m^2$ for some $m \in \mathbb{N}$.

If $n = m^2$ then $\sqrt{n} = m \in \mathbb{Q}$.

Assume $n \neq m^2$. and assume $\sqrt{n} = \frac{p}{q}$ for some $p, q \in \mathbb{N}$.

Then $n = \frac{p^2}{q^2}$, i.e., $nq^2 = p^2$.

Let $p = p_1 \cdots p_r$ & $q = q_1 \cdots q_s$, where $p_1, \ldots p_r, q_1, \ldots, q_s$ are primes.

Then $n(q_1 \cdots q_s^2) = (p_1 \cdots p_r)^2$.

& every $q_1^2, \ldots, q_s^2$ must appear on the right side.

Thus we can ~~d~~ cancel these common factors ~~and~~ on both sides & conclude that

$n = p_{i_1}^2 \cdots p_{i_\ell}^2$. for the left over primes $p_{i_1}, \ldots, p_{i_\ell}$.

~~But~~ It follows that $n = m^2$ with $m = p_{i_1} \cdots p_{i_\ell}$, ~~which~~ which is a contradiction. So $n \neq m^2 \Rightarrow \sqrt{n} \notin \mathbb{Q}$.

(6) (10 points) Define $f : \mathbb{Z}_{49} \to \mathbb{Z}_{49}$ by $f([x]) = [5x]$. Prove that f is a well-defined function and is injective.

Note that if $[x] = [y]$ in $\mathbb{Z}_{49}$ then $x - y = \cancel{49} \, 49k, k \in \mathbb{Z}$

So $5x - 5y = 5 \cdot 49k = 49 \cdot (5k)$.

Hence $[5x] = [5y]$ in $\mathbb{Z}_{49}$. Thus f is well-defined

Suppose $[x], [y] \in \mathbb{Z}_{49}$ s.t. $f([x]) = f([y])$.

Then $[5x] = [5y]$ in $\mathbb{Z}_{49}$,

i.e., $5x - 5y = 49m$, $m \in \mathbb{Z}$

$5(x-y) = 49m = 7^2 m$ ~~[illegible]~~

Now the prime factors $7 \cdot 7$ on the right must be prime factors on the left. So $7 \cdot 7 \mid (x-y)$ i.e., $[x] = [y]$ in $\mathbb{Z}_{49}$. Thus f is injective

(7) (10 points) Define a relation on $\mathbb{R}$ by $(a, b) \in R$ if $a - b$ is an integer.

(a) Prove that R is an equivalence relation.

(b) Describe the elements in the equivalence class $[r]$ for every $r \in [0, 1)$. (Try $[0]$, $[1/2]$, $[1/4]$, etc.)

(c) Show that $\bigcup_{r \in [0,1)} [r] = \mathbb{R}$.

(a) If $a \in \mathbb{R}$, then $a - a = 0 \in \mathbb{Z}$. $\therefore (a,a) \in R$. (Reflexive)

If $a, b \in \mathbb{R}$ s.t. $(a,b) \in R$, then $a - b \in \mathbb{Z}$ and ~~so is~~ $b - a \in \mathbb{Z}$. $\therefore (b,a) \in R$ (Symmetric).

If $a, b, c \in \mathbb{R}$ s.t. $(a,b), (b,c) \in R$, then $a - b \in \mathbb{Z}$, $b - c \in \mathbb{Z}$.

Hence $a - c = (a-b) + (b-c) \in \mathbb{Z}$. (transitive)

(b) $[r] = \{ n + r : n \in \mathbb{Z} \}$

(c) If $x \in \mathbb{R}$, then $x = n + r$ for some integer n & $r \in [0,1)$

Thus $x \in \bigcup_{r \in [0,1)} [r]$. So $\mathbb{R} \subseteq \bigcup_{r \in [0,1)} [r]$.

If $x \in \bigcup_{r \in [0,1)} [r]$, then $x \in [r]$ for some $r \in [0,1)$. Thus $x \in \mathbb{R}$. Hence $\bigcup_{r \in [0,1)} [r] \subseteq \mathbb{R}$.

Combining, we see that $\mathbb{R} = \bigcup_{r \in [0,1)} [r]$.

Rmk: We can also see that for any $x \in \mathbb{R}$, $x = n + r$ for some $n \in \mathbb{Z}$, $r \in [0,1)$ so that $[x] = [r]$, and if $r_1, r_2 \in [0,1)$ then $[r_1] = [r_2]$. So

$[r]$, $r \in [0,1)$ [illegible] $\mathbb{R} = \bigcup [r]$

Case 1° To show f_1 is 1-1. If $f_1(x)=f_1(y)$, $f_1(x)=f_1(y) = -2k$ is even, then $x = y = k$.

Case 2° $f_1(x)=f_1(y) = -2k+1$ is odd, then $x = y = -k$.

Here f_2 is clearly injective.

For f_1, if $n \in \{0,1,2,\dots\}$ then $f(n) = -2n \in \{0,-1,-2,\dots\}$; if $n \in \{-1,-2,\dots\}$ then $f(n) = 1+2n \in \{0,-1,-2,\dots\}$. So it is well-defined.

(8) Show that $|\mathbb{R}| = |\mathbb{R} - \mathbb{N}|$.

[You may construct a bijection, or construct two injections using Schroder-Berstein Theorem.]

Method 1 Let $\mathbb{R} = \mathbb{Z} \cup (\mathbb{R}-\mathbb{Z})$ and $\mathbb{R}-\mathbb{N} = \{0,-1,-2,-3,\dots\} \cup (\mathbb{R}-\mathbb{Z})$.

Define $f_1 : \mathbb{Z} \to \{0,-1,-2,-3,\dots\}$ by $f_1(n) = \begin{cases} -2n & \text{if } n \in \{0\}\cup\mathbb{N} \\ 1+2n & \text{if } n \in \{-1,-2,-3,\dots\} \end{cases}$

and $f_2 : (\mathbb{R}-\mathbb{Z}) \to (\mathbb{R}-\mathbb{Z})$ by $f_2(x) = x$. Then f_1, f_2 are bijections.

Hence $f : \mathbb{R} \to \mathbb{R}-\mathbb{N}$ defined by $f(x) = \begin{cases} f_1(x) & x \in \mathbb{Z} \\ f_2(x) & x \in \mathbb{R}-\mathbb{Z} \end{cases}$ is a bijection.

Method 2. Define $f_1 : \mathbb{R} \to (\mathbb{N} \cup \mathbb{R})$ by $f_1(x) = x$.
Define $f_2 : \mathbb{R} \to (\mathbb{R}-\mathbb{N})$ by $f_2(x) = \frac{x}{1+|x|}$. Then f_1, f_2 are injections.

By Schroder-Berstein Theorem, $|\mathbb{R}| = |\mathbb{R}-\mathbb{N}|$.

Here $f_2(\mathbb{R}) = (-1,1)$ and by homework problem $\hat{f}_2 : \mathbb{R} \to (-1,1)$ defined by $\hat{f}_2(x) = \frac{x}{1+|x|}$ is bijective, so f is injective.

~~Method 3 Define $f : \mathbb{R} \to (\mathbb{R}-\mathbb{N})$ by $f(x) = \frac{x}{1+|x|}$. $f(\mathbb{R}) = (-1,1)$ $\aleph_0$ $|\mathbb{R}| \leq |\mathbb{R}-\mathbb{N}|$ None $|\mathbb{R}-\mathbb{N}| \leq$~~

(9) Let A, B, C, D be non-empty sets such that $|A| = |B|$ and $|C| = |D|$. Show that $|A \times C| = |B \times D|$.

[Hint: Show that there is a bijection from $A \times C$ to $B \times D$.]

Let $|A| = |B|$ and $|C| = |D|$. There the are bijections $f : A \to B$ and $g : C \to D$. Define $h : A \times C \to B \times D$ by $h(a,c) = (f(a), g(c)) \in B \times D$ for any $(a,c) \in A \times C$.

Then $(a,c) \in A \times C \Rightarrow h(a,c) \in B \times D$. So h is well-defined.

If $h(a_1,c_1) = h(a_2,c_2)$ then $(f(a_1), g(c_1)) = (f(a_2), g(c_2))$ i.e., $f(a_1)=f(a_2)$, $g(c_1)=g(c_2)$.

Because f, g are injective, $a_1 = a_2$, $c_1 = c_2$. $\therefore (a_1,c_1) = (a_2,c_2) \in A \times C$. Hence h is injective.

For any $(b,d) \in B \times D$, because f, g are surjective, there are $a \in A$, $c \in C$ s.t. $f(a) = b$, $g(c) = d$. So $h(a,c) = (b,d)$. Hence h is surjective.

Question	1	2	3	4	5	6	7	8	9	Total
Score										

Combining h is bijective. So $|A \times C| = |B \times D|$.