MATH 214 EXAM 1 Name: Sample Solution

To get full credit, you have to justify your answer carefully. Partial credit will be given only if your attempt has made significant progress on the problem.

1. (25 points, 5 points each) Give brief and complete answers to the following:

(a) Construct a subset of $\mathcal{P}(\mathbf{N})$ with 5 elements.

$\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$.

(b) State the contrapositive of the following statement:

If n is an odd integer, then n^2 is divisible by 3 or n^3 is divisible by 3.

If n^2 is not divisible by 3 and n^3 is not divisible by 3 then n is even.

(c) State the negation of the following statement for a given sequence $\{a_1, a_2, \ldots,\}$:
"For every positive number r, there exists an integer $N > 0$ so that for any integer $n > N$ we have $a_n > r$."

There is a positive number r such that for every integer $N > 0$ there is $n > N$ such that $a_n \leq r$.

(d) Prove or disprove the following statement:

$$\exists x \in \mathbf{Z}, \forall y \in \mathbf{Z}, x - y > 0$$

The statement is false.

For any $x \in \mathbb{Z}$, we can let $y = x + 1 \in \mathbb{Z}$ so that $x - y = -1 < 0$.

(e) Show that $T = [0, 3)$ is not a well ordered set.

Consider the subset $S = (0, 3) \subseteq T$

Then S is non-empty. For any $x \in S$, $\frac{x}{2} \in S$. and $\frac{x}{2} < x$. So S has no least element.

2. (12 points) Let A, B be sets. Show that $(A - B) \cup (A \cap B) = A$.

$(A-B) \cup (A \cap B) = (A \cap \bar{B}) \cup (A \cap B)$ because $A-B = A \cap \bar{B}$

$= A \cap (\bar{B} \cup B)$ because of distributive law

$= A \cap U$ $\quad \bar{B} \cup B = U$ is the universal set

$= A$.

3. (12 points) Let $n \in \mathbf{Z}$. Prove that $3 \nmid (2n^2 + 1)$ if and only if $3|n$.

($\Leftarrow$) Suppose $3 | n$, i.e., $n = 3k$, $k \in \mathbb{Z}$.

Then $2n^2+1 = 18k^2+1 \equiv 1 \pmod 3$.

$\therefore$ $2n^2+1$ is not divisible by 3

($\Rightarrow$) We prove the contrapositive: If $3 \nmid n$, then $3 | (2n^2+1)$.

Suppose $3 \nmid n$. Then $n = 3k+1$ or $3k+2$

If $n = 3k+1$, then $2n^2+1 = 2(9k^2+6k+1)+1 = 3(6k^2+4k+1)$.

If $n = 3k+2$, then $2n^2+1 = 2(9k^2+6k+4)+1 = 3(6k^2+4k+3)$

In both cases, $2n^2+1$ is divisible by 3.

4. (12 points) Prove that $n^2 \leq 2^n$ for for every nonnegative integer $n \geq 4$.

Proof by induction:

$P(4)$: $4^2 = 16 = 2^4$ $\therefore$, $P(4)$ holds.

Assume $P(k)$ holds, i.e., $k^2 \leq 2^k$, $k \geq 4$.

Then
$$\begin{aligned}
2^{k+1} &= 2 \cdot 2^k \geq 2 \cdot k^2 && \text{by induction assumption} \\
&= k^2 + k^2 \\
&\geq k^2 + 4k && \because k \geq 4 \\
&\geq k^2 + 2k + 1 \\
&= (k+1)^2 && \therefore P(k+1) \text{ holds.}
\end{aligned}$$

By PMI, $P(n)$ holds for all $n \geq 4$.

5. (12 points) Let x be a nonzero rational number, and y be an irrational number. Show that $x - y$ and y/x are irrational.

Suppose $x = \frac{m}{n} \neq 0$ so that $m, n \in \mathbb{Z}$, $m \neq 0$, $n \neq 0$.

Assume $y \neq \frac{a}{b}$ for any $a, b \in \mathbb{Z}$ with $b \neq 0$.

Consider $x - y$. By contradiction, assume that $x - y \in \mathbb{Q}$

Then $x - y = \frac{c}{d}$ for some $c, d \in \mathbb{Z}$ with $d \neq 0$

So $y = x - \frac{c}{d} = \frac{m}{n} - \frac{c}{d} = \frac{md - cn}{nd} \in \mathbb{Q}$

which is a contradiction. $\boxed{\text{So } x - y \notin \mathbb{Q}}$

Next consider y/x. By contradiction, assume that $y/x \in \mathbb{Q}$

Then $y/x = \frac{f}{g}$ for some $f, g \in \mathbb{Z}$ with $g \neq 0$.

So $y = x \cdot \frac{f}{g} = \frac{mf}{ng} \in \mathbb{Q}$

which is a contradiction. $\boxed{\text{So } y/x \notin \mathbb{Q}.}$

6. (12 points) For $\alpha \geq 0$, let $S_\alpha = [\alpha, \alpha+2)$. Prove that $\cap_{\alpha\in(0,1)} S_\alpha = [1, 2]$.

First we prove $[1,2] \subseteq \bigcap_{\alpha\in(0,1)} S_\alpha$.

Let $x \in [1,2]$, i.e., $1 \leq x \leq 2$.

Then $\alpha < 1 \leq x \leq 2 < 2+\alpha$ for any $\alpha \in (0,1)$.

i.e., $x \in [\alpha, \alpha+2)$ for all $\alpha \in (0,1)$

Hence $x \in \bigcap_{\alpha\in(0,1)} S_\alpha$.

Next we consider $\bigcap_{\alpha\in(0,1)} S_\alpha \subseteq [1,2]$.

We need to show

If $x \in \bigcap_{\alpha\in(0,1)} S_\alpha$ then $x \in [1,2]$.

We prove the contrapositive: If $x \notin [1,2]$, then $x \notin \bigcap_{\alpha\in(0,1)} S_\alpha$

Let $x \notin [1,2]$.

$\boxed{\text{Case 1}^\circ}$ Assume $x < 1$.

We want α_0 such that $x \quad \alpha_0 \quad 1 \quad \alpha_0+2$ so that $x \notin [\alpha_0, \alpha_0+2)$

Choose $\alpha_0 = \begin{cases} \frac{1+x}{2} & \text{if } x > 0, \\ \frac{1}{3} & \text{if } x \leq 0. \end{cases}$

If $1 > x > 0$, then $\alpha_0 = \frac{1+x}{2} \in (0,1)$ and $x \notin [\alpha_0, \alpha_0+2) = S_{\alpha_0}$ because $x < \frac{1+x}{2} = \alpha_0$.

If $x \leq 0$, then $\alpha_0 = \frac{1}{3}$ and $x \notin [\frac{1}{3}, \frac{1}{3}+2) = S_{\alpha_0}$.

In both cases, we see that $x \notin S_{\alpha_0}$ with $\alpha_0 \in (0,1)$

So $x \notin \bigcap_{\alpha\in(0,1)} S_\alpha$.

$\boxed{\text{Case 2}^\circ}$ Assume $x > 2$.

We want α_0 such that $\alpha_0 \quad 2 \quad \alpha_0+2 \quad x$ $\qquad \alpha_0+2 = \frac{2+x}{2}$

Choose $\alpha_0 = \begin{cases} \frac{x-2}{2} & \text{if } x < 4, \\ \frac{1}{2} & \text{if } x \geq 4. \end{cases}$

If $2 < x < 4$, then $\alpha_0 = \frac{x-2}{2} \in (0,1)$ and $x \notin [\alpha_0, \alpha_0+2)$ because $\alpha_0+2 = \frac{x+2}{2} < x$

If $x \geq 4$, then $\alpha_0 = \frac{1}{2} \in (0,1)$ and $x \notin [\frac{1}{2}, 2\frac{1}{2})$

In both cases. $x \notin S_{\alpha_0}$, $\alpha_0 \in (0,1)$