# Math 214 Sample Exam 1 with Solution

1. (25 points, 5 points each) Give brief and complete answers to the following:

   (a) Let $A = \{a, b, c\}$ and $B = \{\emptyset, \{\emptyset\}\}$. Find $A \times B$ and $\mathcal{P}(B)$ (power set of $B$).
   Solution. $A \times B = \{(a, \emptyset), (b, \emptyset), (c, \emptyset), (a, \{\emptyset\}), (b, \{\emptyset\}), (c, \{\emptyset\})\}$; $\mathcal{P}(B) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, B\}$.

   (b) State the contrapositive of the following statement: "If $n$ is an odd integer, then $n^3$ is divisible by 3."
   Solution. If $n$ is an integer such that $n^3$ is not divisible by 3, then $n$ is even.

   (c) State the negation of the following statement:
   "For any $a \geq 0$, there exists $b$ with $0 \leq b \leq 1$, so that for any $c < 0$, $ab = c$."
   Solution. There is $a \geq 0$ such that for every $b \in [0, 1]$ there exists $c < 0$ satisfying $ab \neq c$.

   (d) Prove or disprove the following statement: There exists an integer $x$ so that $x^2 \equiv 3 \pmod 4$.
   Solution. The statement is false. Consider two cases. If $x = 2k$ even, then $x^2 \equiv 4k^2 \equiv 0 \pmod 4$; if $x = 2k + 1$ is odd, then $x^2 \equiv 4k^2 + 4k + 1 \equiv 1 \pmod 4$. Thus, there is no integer $x$ such that $x^2 \equiv 3 \pmod 4$.

   (e) Let $P$ and $Q$ be two statements. Construct a truth table for $P \wedge (Q \Rightarrow (\sim P))$.
   Solution.

| $P$ | $Q$ | $\sim P$ | $(Q \Rightarrow (\sim P))$ | $P \wedge (Q \Rightarrow (\sim P))$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ |

   A different solution. If $P$ is false, then the statement is false. If $P$ is true and $Q$ is true, the statement is false. So, the only case for the statement to be true is when $P$ is true, and $Q$ is false.

2. (15 points) Let $A, B$ be sets. Show that $A \cup B = B$ if and only if $A \subseteq B$.

   Solution. Assume $A \cup B = B$. Then $A \subseteq A \cup B = B$.

   Assume that $A \subseteq B$. Then $A \cup B \subseteq B \cup B = B$ and $B \subseteq A \cup B$; hence $A \cup B = B$.

3. (15 points) Show that $\sqrt[3]{2} = 2^{1/3}$ is irrational. [You cannot use the Fundamental Theorem of Arithmetic.]

   Solution. Proof by contradiction. Assume that $2^{1/3} = m/n$ for some $m, n \in \mathbf{N}$ so that $\gcd(m, n) = 1$. Then $2n^3 = m^3$. Then $m$ cannot be odd. Else, $m^3$ is odd. So, $m = 2k$ for some $k \in \mathbf{N}$. It follows that $2n^3 = 8k^3$ so that $n^3 = 4k^3$. Again, $n$ cannot be odd. Else, $n^3$ is odd. So, $n$ is also even as $m$ is, which contradicts the fact that $\gcd(m, n) = 1$.

4. (15 points) Prove that $8|(7^{2n} - 1)$ for every nonnegative integer $n$.

   Solution. When $n = 0$, we have $8|0$. The statement is true.

   Assume the statement holds for $n = k$, i.e., $7^{2k} - 1 = 8m$ for some $m \in \mathbf{Z}$. Then for $n = k + 1$,

$$7^{2(k+1)} - 1 = 49 \cdot 7^{2k} - 1 = 49(8m + 1) - 1 = 8(49m + 6),$$

   which is a multiple of 8. Thus, the statement also holds for $n = k + 1$.

   By the principle of MI, the statement holds for all nonnegative integer $n$.

5. (15 points) Show that for any $n \in \mathbf{N}$, $n^2$ cannot be of form $5m + 2$ or $5m + 3$, where $m$ is an integer.

   Solution. Suppose $n = 5m + r$ with $r = 0, 1, 2, 3, 4$. Then $n^2 = 25m^2 + 10mr + r^2 \equiv s \pmod 5$ for $s = 0, 1, 4, 4, 1$ depending on $r = 0, 1, 2, 3, 4$. The result follows.

6. (15 points) Suppose $S_\alpha = (-1 - \alpha, 1 + \alpha)$ for $\alpha > 0$. Prove that $\cap_{\alpha \in (0,1)} S_\alpha = [-1, 1]$.

   Solution. Note that $[-1, 1] \subseteq (-1 - \alpha, 1 + \alpha)$ for all $\alpha \in (0, 1)$. Thus, $[-1, 1] \subseteq \cap_{\alpha \in (0,1)} S_\alpha$. To prove the reverse inclusion, suppose $x \in \cap_{\alpha \in (0,1)}$. We show that $x \in [-1, 1]$. If it is not true, then $|x| > 1$. Let $\beta = \min\{(|x| - 1)/2, 1/2\} \in (0, 1)$. Then $|x| = 2\beta + 1 > \beta + 1$ so that $x \notin S_\beta$. So, $x \notin \cap_{\alpha \in (0,1)} S_\alpha$.