Math 214 Final Examination **Your Name:** ____________________

Ten points for each question except for Problem 1. Good Luck!

(1) (Five points for each part.)

(a) Write the negation of the following statement:
"For any $a \geq 0$, there exists $n \in \mathbb{N}$ such that $1/n < a$."

There exists $a \geq 0$ such that for all $n \in \mathbb{N}$ we have $\frac{1}{n} \geq a$.

(b) Prove or disprove the following. There is $n \in \mathbb{Z}$ such that $n^2 = 7k + 3$ for some $k \in \mathbb{Z}$.

False: Let $n \in \mathbb{Z}$. Then $n = 7q + r$ with $r = 0, 1, 2, 3, 4, 5, 6$.

So $[n^2] = [r^2] = [0], [1], [4], [2], [2], [4], [1]$ in $\mathbb{Z}_7$

Hence $[n^2] \neq [3]$, i.e., $n^2 \neq 7k + 3$.

(c) Prove or disprove the following. If $A \subseteq B$, and if B is denumerable, then A is denumerable.

False: Let $A = \{1\}$, $B = \mathbb{N}$.

Then $A \subseteq B$, B is denumerable but A is not.

(d) Consider the equivalence relation R on $\mathbb{R}^2$ defined by $((x_1, y_1), (x_2, y_2)) \in R$ if $x_1^2 + y_1^2 = x_2^2 + y_2^2$. Describe geometrically the equivalence class for $(1, 0)$.

$[(1,0)] = \{(x,y) : x^2 + y^2 = 1^2 + 0^2\} = \{(x,y) : x^2 + y^2 = 1\}$.

So it is the circle in $\mathbb{R}^2$ centered at $(0,0)$ with radius 1.

(2) Use the Euclidean Algorithm to find $d = \gcd(346, 123)$, and find TWO pairs of points $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ so that $2d = 346x + 123y$.

$346 = 2 \cdot (123) + 100$
$123 = 1 \cdot (100) + 23$
$100 = 4 \cdot (23) + 8$
$23 = 2 \cdot (8) + 7$
$8 = 1 \cdot (7) + 1$

Thus $\gcd(346, 123) = 1 = d$

Now, $1 = 8 - 1 \cdot 7$
$= 8 - (23 - 2 \cdot 8)$
$= 3 \cdot 8 - 23$
$= 3(100 - 4 \cdot 23) - 23$
$= 3 \cdot 100 - 13 \cdot 23$
$= 3 \cdot 100 - 13(123 - 100)$
$= 16 \cdot 100 - 13 \cdot 123$
$= 16(346 - 2 \cdot 123) - 13 \cdot 123$

$\therefore d = 1 = 16 \cdot 346 - 45 \cdot 123$

$2d = 32 \cdot 346 - 90 \cdot 123$

$= (32 + (123)) \cdot 346 - (90 + 346) 123.$

So $(x, y) = (32, -90)$
& $(x, y) = (155, -436)$
are two pairs of integers satisfying the requirement.

(3) Prove the following. (a) $\sqrt{2016}$ is irrational. (b) $\log(2016)$ is irrational.

Note: $\boxed{2016 = 2^5 \cdot 7 \times 3^2}$

(a) If $\sqrt{2016} = \frac{p}{q}$ ($p, q \in \mathbb{N}$) then $2016 q^2 = p^2$

Thus $2^5 \cdot 7 \cdot 3^2 q^2 = p^2$.

Consider the number of the prime factor 7 on both sides. It will occur in an odd number of times on the left but an even number of times on the right, which is impossible.

(b) If $\log(2016) = \frac{p}{q}$ ($p, q \in \mathbb{N}$), then $\log(2016)^q = p$

Then $(2016)^q = 10^p = 2^p 5^q$.

5 is a prime factor on the right but not a prime factor on the left, which is impossible.

(4) Suppose $a, b \in \mathbb{N}$ are distinct odd numbers and none of them is divisible by 3. Show that $a^2 - b^2$ is divisible by 24.

Let $n = a^2 - b^2$.

Because $a \neq 3p$, $b \neq 3q$, $[a^2] = [1]$ in $\mathbb{Z}_3$, $[b^2] = [1]$ in $\mathbb{Z}_3$.

So $[a^2 - b^2] = [0]$ in $\mathbb{Z}_3$, i.e., 3 is a factor of n.

Now $n = \cancel{(2k+1)^2}$. let $a = 2k+1$, $b = 2m+1$

Then $a^2 - b^2 = 4[k^2+k - m^2 - m]$.

Note That $k^2+k = k(k+1)$ is even

& $m^2+m = m(m+1)$ is even.

So $\frac{n}{4} = [k^2+k - m^2 - m)$ is even. Thus $a^2 - b^2$ is divisible by 8

Hence, $n = 3 \cdot 2^3 \cdot g$ for some $g \in \mathbb{Z}$. i.e., $24 \mid n$

(5) Let $a_0 = 0$, $a_1 = 1$, and $a_n = 2a_{n-1} - a_{n-2} + 2$ for $n \geq 2$. Make a conjecture on the general form of a_n in terms of n and prove your conjecture using induction.

$P(n)$: $a_n = n^2$.

The result holds for $n = 0, 1$.

Assume the result hold for all $k = 1, \cdots, m$, $m \geq 1$.

Then

$$
\begin{aligned}
a_m &= 2a_{m-1} - a_{m-2} + 2 \\
&= 2(m-1)^2 - (m-2)^2 + 2 \\
&= 2m^2 - 4m + 2 - (m^2 - 4m + 4) + 2 \\
&= m^2
\end{aligned}
$$

So $P(m)$ holds.

By the principle of MI, $P(n)$ holds for all $n \in \{0\} \cup \mathbb{N}$.

(6) Let R_1 and R_2 be two relations on a non-empty set A. Prove or disprove the following.

(a) If $R_1 \cup R_2$ is reflexive, then so are R_1 and R_2. False

(b) If $R_1 \cup R_2$ is symmetric, then so are R_1 and R_2. False

(c) If $R_1 \cup R_2$ is transitive, then so are R_1 and R_2. False

Let $A = \{1, 2, 3\}$

(a) $R_1 = \{(1,1), (1,2), (2,3), (3,1)\}$

$R_2 = \{(2,2), (3,3), (2,1), (3,2), (1,3)\}$

Then $R_1 \cup R_2$ is an equivalence relation so that it is reflexive, symmetric & transitive.

But ~~R_1, R_2~~ R_1 is not reflexive $\because (2,2) \notin R_1$

R_1 is not symmetric $\because (1,2) \in R_1$, but $(2,1) \notin R_1$.

R_1 is not transitive $\because (1,2), (2,3) \in R_1$ but $(1,3) \notin R_1$.

R_2 is not reflexive $\because (1,1) \notin R_2$

R_2 is not symmetric $\because (2,1) \in R_2$ but $(1,2) \notin R_2$

R_2 is not transitive $\because (3,2), (2,1) \in R_2$ but $(3,1) \notin R_2$.

(7) Let S and T be two sets. Prove that if $|S - T| = |T - S|$, then $|S| = |T|$.

Let $f_1 : S \cap T \to S \cap T$ be defined by $f(x) = x$

Let $f_2 : S - T \to T - S$ be a bijection.

Now $S = (S \cap T) \cup (S - T)$ a disjoint union, $T = (S \cap T) \cup (T - S)$, a disjoint union

Then define $f : S \to T$ by

$$f(x) = \begin{cases} x & \text{if } x \in S \cap T, \\ f_2(x) & \text{if } x \in S - T. \end{cases}$$

will be a bijection.

Well-defined: One can also check that $x \in S \Rightarrow f(x) \in T$. So f is well-defined.

1-1: If $f(x) = f(y)$. then (a) $f(x) = f(y) \in S \cap T \Rightarrow x = y \in S \cap T$

(b) $f(x) = f(y) \in T - S \Rightarrow f(x) = f_2(x)$, $f(y) = f_2(y)$ with $x, y \in S - T$.

So $x = y$.

Onto: For any $y \in T$. If $y \in S \cap T$ then $f(y) = y$; If $y \in T - S$, there is $x \in S - T$

(8) Show that the set of functions from $\{1, 2, 3\}$ to $\mathbb{N}$ is denumerable.

[You may use the facts: (a) $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ is denumerable, (b) a subset of a denumerable set is finite or denumerable.] Let S be the set of functions from $\{1,2,3\}$ to $\mathbb{N}$.

Note that we can define

$$F: S \to \mathbb{N} \times \mathbb{N} \times \mathbb{N} \text{ by}$$

~~Th~~ $F(f) = (f(1), f(2), f(3))$.

Then F is well-defined because every $f \in S$ will yield $F(f) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$

It is 1-1 because $F(f) = F(g)$ implies $(f(1), f(2), f(3)) = (g(1), g(2), g(3))$, $\therefore f = g$

It is onto because for any $(n_1, n_2, n_3) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}$, we have $f: \{1, 2, 3\}$ s.t. $(f(1), f(2), f(3)) = (n_1, n_2, n_3)$ so that $F(f) = (n_1, n_2, n_3)$.

Thus $|S| = |\mathbb{N} \times \mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$

(9) For nonempty sets A and B and functions $f : A \to B$ and $g : B \to A$, suppose that $f \circ g = 1_B$, the identity function on B. Prove that f is injective if and only if g is surjective.

Assume $f: A \to B$, $g: B \to A$, $f \circ g = 1_B$, the identity map.

If f is not injective then there are $a_1 \neq a_2$ in A s.t. $f(a_1) = f(a_2) = b$. If ~~$g(x) = b$~~ g is surjective so that

~~Thus~~ $g(b_1) = a_1$ & $g(b_2) = a_2$, then

$b_1 = f \circ g(b_1) = f(a_1) = b$ & $b_2 = f \circ g(b_2) = f(a_2) = b$.

So $g(b) = a_1$ & $g(b) = a_2$, which is a contradiction. Hence g is not surjective.

Now suppose f is injective.

If g is not surjective, then there is $a \in A$ s.t. $g(b) \neq a$ for any $b \in B$.

Now ~~$f(\hat{a}) = \hat{b} \in$~~ $f(a) = \hat{b} \in B$. & $\hat{b} = f(g(\hat{b})) = f(\hat{a})$.

We must have $f(\hat{a}) = f(a)$. but $g(\hat{b}) \neq a$. So f is not injective, a contradiction

(10) Let $\mathcal{P}(A)$ be the power set of a non-empty set A. Show that $|A| < |\mathcal{P}(A)|$.

Clearly $g: A \to \mathcal{P}(A)$ defined by $g(a) = \{a\}$ is an injection.

Suppose $f: A \to \mathcal{P}(A)$ is a surjection.

Then consider $T = \{a \in A : a \notin f(a)\}$

We claim that there is no $t \in A$ s.t. $f(t) = T$.

Case 1: If $t \in T$, then $t \in T = f(t)$, contradicting. By the definition of T, ~~$t \notin T$, a contradiction~~

Case 2: If $t \notin T$, then $t \notin T = f(t)$ implies that $t \in T$ by the definition of T, which is again a contradiction.

Thus there is no $t \in A$ s.t. $f(t) = T$.

Question	1	2	3	4	5	6	7	8	9	10	Total
Score											