Math 214 – Foundations of Mathematics **Homework 1** Sample Solution.

1. Write each of the following sets as specified.

   (a) List the elements in the set $A = \{n \in \mathbf{N} : n^3 < 1000\}$.

   (b) Describe the set $B = \{-2, -1, 0, 1, 2, 3, 4\}$ using the notation $\{n : p(n)\}$, where $p(n)$ specifies the property of element n.

   Answer: (a) $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

   (b) $B = \{n \in \mathbf{Z} : -3 < n < 5\}$.

2. Recall that for a set A, $\mathcal{P}(A)$ denotes the power set of A.

   (a) Find $\mathcal{P}(\mathcal{P}(\{a\}))$ and its cardinality.

   (b) Give an example of a set S such that $S \in \mathcal{P}(\mathbf{N})$ and $|S| = 6$.

   (c) Give an example of a set S such that $S \subseteq \mathcal{P}(\mathbf{N})$ and $|S| = 6$.

   Answer: (a) $\mathcal{P}(\{a\}) = \{\emptyset, \{a\}\}$; $\mathcal{P}(\mathcal{P}(\{a\})) = \{\emptyset, \{\emptyset\}, \{\{a\}\}, \{\emptyset, \{a\}\}\}$ has cardinality 4.

   (b) $S = \{1, 2, 3, 4, 5, 6\}$.

   (c) $S = \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$.

3. The following problems involve set operations.

   (a) Give an example of three non-empty sets A, B, and C such that $B \neq C$ but $B - A = C - A$.

   (b) Let $A = \{\emptyset, \{\emptyset\}\}$. Find $\mathcal{P}(A) - A$.

   Answer: (a) Let $A = \{1, 2\}$, $B = \{1\}$, and $C = \{2\}$. Then $B - A = C - A = \emptyset$.

   (b) $\mathcal{P}(A) - A = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, A\} - A = \{\{\{\emptyset\}\}, A\}$.

4. For a real number r, define S_r to be the interval $[r-1, r+2)$. Let $A = \{1, 3, 4\}$. Determine $\cup_{\alpha \in A} S_\alpha$ and $\cap_{\alpha \in A} S_\alpha$.

   (a) List the intervals S_r for $r \in A$.

   (b) Determine $\cup_{\alpha \in A} S_\alpha$ and $\cap_{\alpha \in A} S_\alpha$.

   (c) (Extra 3 points) Determine $\cap_{\alpha \in (0,1)} S_\alpha$ and $\cup_{\alpha \in (0,1)} S_\alpha$.

   Solution. (a) $S_1 = [0, 3), S_3 = [2, 5), S_4 = [3, 6)$.

   (b) $\cup_{\alpha \in A} S_\alpha = S_1 \cup S_3 \cup S_4 = [0, 6)$; $\cap_{\alpha \in A} S_\alpha = S_1 \cap S_3 \cap S_4 = \emptyset$.

   (c) $\cap_{\alpha \in (0,1)} S_\alpha = [0, 2]$ and $\cup_{\alpha \in (0,1)} S_\alpha = (-1, 3)$.

5. For two sets A and B, recall that $A \times B$ is the Cartesian product of A and B.

   (a) Let $A = \{a, b\}$. Determine $A \times \mathcal{P}(A)$.

   (b) Let $A = \{0, 1\}$ and $B = [0, 2] \cap [1, 3]$. Depict or describe geometrically the set $A \times B$ in $\mathbf{R}^2$.

   (c) Let $A = \{0, 1\}$, $B = (0, 1) \cap A$ and $C = \mathbf{R}$. What is $A \times B \times C$?

Answer. (a) $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. So,

$$A \times \mathcal{P}(A) = \{(a, \emptyset), (a, \{a\}), (a, \{b\}), (a, \{a, b\}), (b, \emptyset), (b, \{a\}), (b, \{b\}), (b, \{a, b\})\}.$$

(b) $A \times B = L_1 \cup L_2$, where $L_1 = \{(0, b) : 1 \leq b \leq 2\}$ and $L_2 = \{(1, b) : 1 \leq b \leq 2\}$.

Geometrically, L_1 is the line on the $x - y$ plane joining the points $(0, 1)$ and $(0, 2)$, and L_2 is the line on the $x - y$ plane joining the points $(1, 1)$ and $(1, 2)$, and all the endpoints of the line segments are included.

(c) Note that $B = \emptyset$. So, there is no triple of the form (a, b, c) with $a \in A, b \in B, c \in C$. Thus, $A \times B \times C = \emptyset$.

6. Determine all different partitions of the set $\{1, 2, 3\}$.

Answer. The set of partitions of $\{1, 2, 3\}$ equals $\{P_1, \ldots, P_5\}$, where $P_1 = \{\{1\}, \{2\}, \{3\}\}$; $P_2 = \{\{1, 2\}, \{3\}\}$; $P_3 = \{\{1\}, \{2, 3\}\}$; $P_4 = \{\{1, 3\}, \{2\}\}$; $P_5 = \{\{1, 2, 3\}\}$.

Extra credit problem. If a set A has n elements, show that $\mathcal{P}(A)$ has 2^n elements.

Solution. Suppose $A = \{a_1, \ldots, a_n\}$ has cardinality n. To form a subset, we have to decide whether the subset contains or not contain a_k for $k = 1, \ldots, n$. For each a_k there are two possible answers: Yes or No. So, there are 2^n possible different answers for the n questions. In particular, if we say no to each a_k, then we get the empty set; if we say yes to each a_k, then we get the whole set A. Thus, each of the 2^n different answers for the n questions will give rise to a different subset. So, A has 2^n subsets.