Math 214 Homework 2 **Sample solution**

1. (4 points) Let P: 15 is odd, Q: 21 is prime and R: $\frac{1}{2} \in \mathbb{N}$. State each of the following in words, and determine whether they are true or false.

   (a) $P \vee Q$ Answer: 15 is odd or 21 is prime. True

   (b) $P \wedge Q$ Answer: 15 is odd and 21 is prime. False

   (c) $(\sim P) \vee Q$ Answer: 15 is even or 21 is prime. False

   (d) $P \wedge (\sim Q \vee R)$ Answer: 15 is odd, and that either 21 is not prime or $\frac{1}{2} \in \mathbb{N}$. True

2. (4 points) In each of the following, two open sentences $P(x)$ and $Q(x)$ over a domain S are given. For each part, determine $T = \{x \in S : \text{“}P(x) \Rightarrow Q(x)'' \text{ is true}\}$.

   (a) $P(x) : x - 3 = 4; Q(x) : x \geq 8; S = \mathbb{R}$.
   Answer: Here $T = \{x \in \mathbb{R} : x \neq 7\} = \mathbb{R} - \{7\}$.
   Reason: For $x = 7$, $P(x)$ is true, $Q(x)$ is false, so “$P(x) \Rightarrow Q(x)$” is false. Thus, $7 \notin T$. For $x \neq 7$, $P(x)$ is false; so “$P(x) \Rightarrow Q(x)$” is true. Thus $x \in T$.

   (b) $P(x) : x \in [-1, 2]; Q(x) : x^2 \leq 2; S = [-1, 1]$.
   Answer: $T = [-1, 1]$.
   Reason. For any $x \in [-1, 1]$, $P(x)$ is true and $Q(x)$ is true. Thus $P(x) \Rightarrow Q(x)$ is true for all $x \in [-1, 1]$.

3. (4 points) Let P, Q be statements. Show that $\sim (P \implies Q)$ and $P \wedge (\sim Q)$ are logically equivalent using truth table.

   Solution. The truth tables for the two statements are:

P	Q	$P \implies Q$	$\sim (P \implies Q)$	$P \wedge (\sim Q)$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	F	F

   Thus, the statements $\sim (P \implies Q)$ and $P \wedge (\sim Q)$ are logically equivalent.

4. (6 points) Write the statements so that there are no $\sim$ symbols. Then, rewrite the statements so that there are no $\forall, \exists, \in$ or $=$ symbols.

   (a) $\sim (\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, xy = 1)$;
   Answer: This statement is equivalent to $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, xy \neq 1$. In English, this statement is:
   There is a real x such that for all real y, the product of x and y is not one.

   (b) $\sim (\exists y \in \mathbb{R}, \forall x \in \mathbb{R}, xy = 0)$;
   Answer: This statement is equivalent to $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, xy \neq 0$. In English, this statement is:
   For all real y there is a real x such that the product of x and y is not zero.

   (c) $\sim (\exists n \in \mathbb{Z}, \exists m \in \mathbb{Z}, m \leq n)$;
   Answer: This statement is equivalent to $\forall n \in \mathbb{Z}, \forall m \in \mathbb{Z}, m > n$. In English, this statement is:
   For any integers n and m, n is strictly less than m.

5. (4 points) Consider the sentence,

   “For every integer $n > 0$ there is some real number $x > 0$ such that $x < 1/n$.”

   (a) Without using words of negation, write a complete sentence that negates the sentence.

   (b) Determine whether the original or the negation of the statement is true with explanation.

   Answer: (a) The negation is: there is a positive integer n such that all positive real numbers x satisfy $x \geq 1/n$.

   (b) The original statement is true. Reason. For every positive integer n, we can set $x = \frac{1}{2n} > 0$ so that $x = \frac{1}{2n} < 1/n$.

6. (4 points) For $\alpha \in \mathbb{R}$, let $S_\alpha = (-\alpha, \alpha)$. Prove or disprove the following statements.

(a) $\forall \alpha \in (0,1), \exists \beta \in (0,1), S_\alpha \subset S_\beta$ (note that $\subset$ and $\subseteq$ are not the same).
Answer: This statement is true.
Proof: For any $\alpha \in (0,1)$, let $\beta = \frac{\alpha+1}{2}$. Then $0 < 2\alpha = \alpha + 1 < 2$. Dividing each term by 2, we see that $0 < \alpha < \beta < 1$. So, $\beta \in (0,1)$ and $S_\alpha \subset S_\beta$.

(b) $\exists \alpha \in (0,1), \forall \beta \in (0,1), S_\alpha \subset S_\beta$.
Answer: This statement is false.
Proof: Suppose $\alpha \in (0,1)$. Let $\beta = \alpha/2 > 0$. Then $\beta \in (0,1)$, but $S_\alpha \not\subset S_\beta$. So, we cannot find $\alpha \in (0,1)$ having the required property.