Math 214 Foundations of Mathematics Homework 3 Sample Solution

1. (4 points) Let A, B, C be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

   Proof 1. Note that

$$\begin{aligned} A-(B\cap C) &= \{x : (x \in A) \text{ and } x \notin (B \cap C)\} \\ &= \{x : (x \in A) \text{ and } (\sim (x \in B \text{ and } x \in C))\} \\ &= \{x : (x \in A) \text{ and } (x \notin B \text{ or } x \notin C)\} \\ &= \{x : (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C)\} \\ &= \{x : x \in (A - B) \text{ or } x \in (A - C)\} \\ &= (A - B) \cup (A - C). \end{aligned}$$

   Proof 2. We first show $(A - B) \cup (A - C) \subseteq A - (B \cap C)$.

   Since $B \cap C \subseteq B$, $(A - B) \subseteq A - (B \cap C)$; also, $B \cap C \subseteq C$ so that $(A - C) \subseteq A - (B \cap C)$. Thus, $(A - B) \cup (A - C) \subseteq A - (B \cap C)$.

   Next, we show that $A - (B \cap C) \subseteq (A - B) \cup (A - C)$. Suppose $x \in A - (B \cap C)$.

   Then, $x \in A$ and $x \notin B \cap C$. By De Morgan's Law, we have

   $(x \in A$ and $x \notin B)$ or $(x \in A$ and $x \notin C)$. Thus, $x \in (A - B) \cup (A - C)$.

2. (4 points) Let A, B, C and D be sets. Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

   Proof: Note that

$$\begin{aligned} (A\times B)\cap(C\times D) &= \{(x,y) : (x,y) \in A \times B \text{ and } (x,y) \in C \times D\} \\ &= \{(x,y) : (x \in A \text{ and } y \in B) \text{ and } (x \in C \text{ and } y \in D)\} \\ &= \{(x,y) : (x \in A \text{ and } x \in C) \text{ and } (y \in B \text{ and } y \in D)\} \\ &= \{(x,y) : x \in A \cap C \text{ and } y \in B \cap D\} \\ &= (A \cap C) \times (B \cap D). \end{aligned}$$

3. (4 points) For the following, state whether they are true or not. Then, prove your answer.

   (a) $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}, xy = 1$;
   False. Let $x = 0$. Then we have $xy = 0$ for any $y \in R$ so that $xy \neq 1$.

   (b) $\exists n \in \mathbf{N}, \exists m \in (\mathbf{N} - \{1\}), nm = 1$;
   False. We show that the negation of the statement if true:
   For any $n \in N$ and $m \in N - \{1\}$, $n \geq 1$ and $m \geq 2$, so $nm \geq 2$.

4. (4 points) Show that for any two positive numbers a and b,

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4.$$

   Proof. Note that

$$(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)-4 = \frac{1}{ab}(a+b)^2-4 = \frac{1}{ab}[(a^2+2ab+b^2)-4ab] = \frac{1}{ab}[(a^2+b^2-2ab] = \frac{1}{ab}(a-b)^2 \geq 0.$$

   So, $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right) \geq 4$.

5. (4 points) Let $m = 4s + 2$ with $s \in \mathbf{Z}$. Show that there are no integers x, y such that

$$x^2 - y^2 = m.$$

Proof. Consider the following cases depending on the parities of x and y.

Case 1. If there are $l, n \in \mathbf{Z}$ such that $x = 2l$ and $y = 2n$ are both even, then $x^2 - y^2 = (2l)^2 - (2n)^2 = 4l^2 - 4n^2 = 4(l^2 - n^2)$, which is divisible by 4 and thus cannot equal $m = 4s+2$.

Case 2. If there are $l, n \in \mathbf{Z}$ such that $x = 2l$ is even and $y = 2n + 1$, then $x^2 - y^2 = (2l)^2 - (2n+1)^2 = 4l^2 - (4n^2 + 4n + 1) = 4(l^2 - n^2 - n) - 1$, which is odd and thus cannot equal $m = 4s + 2$.

Case 3. If there are $l, n \in \mathbf{Z}$ such that $x = 2l + 1$ is odd and $y = 2n$ is even, then $x^2 - y^2 = (2l+1)^2 - (2n)^2 = (4l^2 + 4l + 1) - 4n^2 = 4(l^2 + l - n^2) + 1$, which is odd and thus cannot equal $m = 4s + 2$.

Case 4. If there are $l, n \in \mathbf{Z}$ such that $x = 2l + 1$ is odd and $y = 2n + 1$ is odd, then $x^2 - y^2 = (2l+1)^2 - (2n+1)^2 = (4l^2 + 4l + 1) - (4n^2 + 4n + 1) = 4(l^2 + l - n^2 + n)$, which is divisible by 4 and thus cannot equal $m = 4s + 2$.

6. (4 points) Prove that the product of an irrational number and a nonzero rational number is irrational.

   Proof. Assume the contrary that for some $s \in \mathbf{Q} - \{0\}$ and $t \notin \mathbf{Q}$, $st \in \mathbf{Q}$. Then $s = \frac{p}{q}$ and $st = \frac{x}{y}$ for some $p, q, x, y \in \mathbf{Z}$ and $p, q, x \neq 0$. Then $t = \frac{xq}{yp}$ with $xq, yp \in \mathbf{Z}$ and $yp \neq 0$. So $t \in \mathbf{Q}$, a contradiction.

7. (4 points) Let $S = \{a, b, c\} \subseteq \mathbf{Z}$. For any non-empty subset X of S, let $s(X)$ be the sum of elements in X. Show that there are non-empty subsets A, B of S such that $s(A) - s(B)$ is divisible by 6.
   Proof. There are 7 possible subsets of S that A and B may be: $\{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}$, and $\{a,b,c\}$. Therefore, there are 7 possible sums: $a, b, c, a+b, a+c, b+c$, and $a+b+c$. When dividing by 6, however, there are only 6 possible remainders: 0, 1, 2, 3, 4, and 5. Thus, at least two of the sums must have the same remainder when dividing by 6. Call one of these sums $s(A) = 6n + r$, with $n \in \mathbf{Z}$ and remainder $r \in \{0,1,2,3,4,5\}$, and call the other $s(B) = 6m + r$ with $m \in Z$ and remainder r when divided by 6. Thus $s(A) - s(B) = 6n + r - (6m + r) = 6n - 6m = 6(n - m)$, which will be divisible by 6.

8. (Extra credit. 4 points) Recall that for a given $S \subseteq \mathbf{R}$, the maximum element of S, denoted by $\max S$, is the number $\alpha \in S$ such that $\alpha \geq \beta$ for all $\beta \in S$.

   Let $A = \{n \in \mathbf{N} : \sqrt{n} \notin \mathbf{Q}\}$. Show that $\max A$ does not exist.

   Proof. Suppose on the contrary that $\max A = N$ exists in A. Then $n = 2N^2 \in A$ because $\sqrt{n} = \sqrt{2}N$ is the product of a irrational number $\sqrt{2}$ and a nonzero rational number N. Clearly, $n = 2N^2 > N$, which contradicts the fact that $N = \max A$. Thus, $\max A$ does not exist.