Math 214 – Foundations of Mathematics **Homework 3**

1. (5 points) Let A, B, C be sets. Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

   Hint: You may use one of the following three approaches.

   a) Write $(A - B) \cup (A - C) = \{x \in U : p(x)\}$, where $p(x) : x \in (A - B)$ or $x \in (A - C)$, and $A - (B \cap C) = \{x \in U : q(x)\}$, where $q(x) : x \in A$ and $x \notin (B \cap C)$, where U is the universal set. Show that $p(x)$ and $q(x)$ are logically equivalent.
   b) Show that if $x \in (A - B) \cup (A - C)$, then $x \in A - (B \cap C)$. Also, show that if $x \in A - (B \cap C)$, then $x \in (A - B) \cup (A - C)$.
   c) Use set operations such as $A - X \subseteq A - Y$ if $Y \subseteq X$, to argue $(A - B) \cup (A - C) \subseteq A - (B \cap C)$ and also $A - (B \cap C) \subseteq (A - B) \cup (A - C)$.

2. (5 points) Let A, B, C and D be sets. Prove that

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

   Hint: Show that (a) if $(x, y) \in (A \times B) \cap (C \times D)$, then $(x, y) \in (A \cap C) \times (B \cap D)$, and (b) if $(x, y) \in (A \cap C) \times (B \cap D)$, then $(x, y) \in (A \times B) \cap (C \times D)$.

3. (5 points) For the following, state whether they are true or not. Then, prove your answer.

   (a) $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}, xy = 1$;
   (b) $\exists n \in \mathbf{N}, \exists m \in (\mathbf{N} - \{1\}), nm = 1$.

   Hint: If you want to prove that P is FALSE, you may try to prove $\sim P$ is TRUE.

4. (5 points) Show that for any two positive numbers a and b,

$$(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4.$$

   Hint: Reduce the problem to $(a + b)^2 \geq 4ab$, and use algebra.

5. (5 points) Let $m = 4s + 2$ with $s \in \mathbf{Z}$. Show that there are no integers x, y such that

$$x^2 - y^2 = m.$$

   Hint: Consider 4 cases according to x, y are even or odd.

6. (5 points) Prove that the product of an irrational number and a nonzero rational number is irrational.

   Hint: Assume that x is irrational and y is nonzero rational. If xy is rational, then ...

7. (5 points) Let $S = \{a, b, c\} \subseteq \mathbf{Z}$. For any non-empty subset X of S, let $s(X)$ be the sum of elements in X. Show that there are non-empty subsets A, B of S such that $s(A) - s(B)$ is divisible by 6.

   Hint: For each non-empty subset X of S, consider the remainder of $s(X)$ divided by 6.

8. (Extra Credit, 5 points) Recall that for a given $S \subseteq \mathbf{R}$, the maximum element of S, denoted by $\max S$, is the number $\alpha \in S$ such that $\alpha \geq \beta$ for all $\beta \in S$.

   Let $A = \{n \in \mathbf{N} : \sqrt{n} \notin \mathbf{Q}\}$. Show that $\max A$ does not exist.

   [Hint: First, use Problem 6 to deduce that $2m^2 \in A$ for any positive integer m. Then use this fact to argue that A has no maximum element.]