Math 214 – Foundations of Mathematics Homework 4 Sample Solution

1. To prove $\cap_{\alpha\in(0,1)} S_\alpha = \{0\}$.

   Note that $0 \in (-\alpha,\alpha) = S_\alpha$ for every $\alpha \in (0,1)$. Thus, $\{0\} \subseteq \cap_{\alpha\in(0,1)} S_\alpha$.

   For the reverse inclusion, we need to show that if $x \in \cap_{\alpha\in(0,1)} S_\alpha$ then $x = 0$. We show the contrapositive and consider two cases.

   Case 1. If x is such that $|x| \geq 0.5$, then $x \notin (-0.5, 0.5) = S_{0.5}$ so that $x \notin \cap_{\alpha\in(0,1)} S_\alpha$.

   Case 2. If $0 < |x| < 0.5$, then $x \notin (-x, x) = S_x$ so that $x \notin \cap_{\alpha\in(0,1)} S_\alpha$.

   Next, we prove $\cup_{\alpha\in(0,1)} S_\alpha = (-1,1)$. If $x \in \cup_{\alpha\in(0,1)} S_\alpha$, then $x \in (-\alpha,\alpha) = S_\alpha$ for some $\alpha \in (0,1)$. Thus, $x \in (-1,1)$.

   To prove the reverse inclusion, suppose $x \in (-1,1)$. Let $\beta = (1 + |x|)/2$. Then $|x| \leq \beta$ so that $x \in S_\beta$. Hence, $x \in \cup_{\alpha\in(0,1)} S_\alpha$.

2. If U is the universal set, then

$$
\begin{aligned}
(A - B) \cup (A \cap B) &= (A \cap \overline{B}) \cup (A \cap B) && \text{by the definition of } A - B \\
&= A \cap (\overline{B} \cup B) && \text{by the distributive law} \\
&= A \cap U && U \text{ is the universal set} \\
&= A.
\end{aligned}
$$

3. For any three sets A, B, C, show that $(A \times C) - (B \times C) \subseteq (A - B) \times C$.

   Solution. Suppose $(x, y) \in (A \times C) - (B \times C)$. Then $(x, y) \in A \times C$ and $(x, y) \notin B \times C$. So, $(x \in A, y \in C)$ and it is not true that $(x \in B, y \in C)$. Since we know that $y \in C$, so $x \notin B$. Thus, we have $x \in A, x \notin B, y \in C$. Hence, $x \in (A \cap \overline{B}) = (A - B)$ and $y \in C$, i.e., $(x, y) \in (A - B) \times C$.

   Suppose $(x, y) \in (A-B)\times C$. Then $x \in A, x \notin B, y \in C$. So, $(x, y) \in A\times C$ and $(x, y) \notin B\times C$, i.e., $(x, y) \in (A \times C) - (B \times C)$.

4. Prove that if $a, b, c, d \in \mathbf{R}$, then $(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$.

   Solution. Note that

$$
(a^2 + c^2)(b^2 + d^2) - (ab + cd)^2 = (ab)^2 + (ad)^2 + (cb)^2 + (cd)^2 - (a^2b^2 + 2abcd + c^2d^2)
$$

$$
= (ad)^2 - 2abcd + (bc)^2 = (ad - bc)^2 \geq 0.
$$

   So, $(a^2 + c^2)(b^2 + d^2) \geq (ab + cd)^2$.

5. Prove that if $a, b, c \in \mathbf{R}$, then $|x - z| \leq |x - y| + |y - z|$.

   Solution. Let $a = x - y$ and $b = y - z$. Then $a + b = (x - y) + (y - z) = (x - z)$.

   We need to prove $|a + b| \leq |a| + |b|$. Note that $|p| \leq L$ for a positive number L if and only if $-L \leq p \leq L$. Now, $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. So, $-(|a| + |b|) \leq a + b \leq |a| + |b|$. Thus, $|a + b| \leq |a| + |b|$.

6. Let $n \in \mathbf{Z}$. Prove that $3|(2n^2+1)$ if and only if $3 \not| n$.

   Solution. If $3 \not| n$, then $n = 3k+1$ or $3k+2$. Thus,

$$2n^2+1 = 2(3k+1)^2+1 = 2(9k^2+6k+1)+1 = 3(6k^2+4k+1)$$

$$\text{or} \quad 2n^2+1 = 2(3k+2)^2+1 = 2(9k^2+6k+4)+1 = 3(6k^2+4k+3).$$

   In either case, $2n^2+1$ is divisible by 3.

   To prove the converse (reverse implication), we use the indirect proof, i.e., prove the contrapositive of the statement. Assume that $3|n$ so that $n = 3k$. Then

$$2n^2+1 = 18k^2+1 = 3(6k^2)+1$$

   is not divisible by 3.

7. Let $a, b \in \mathbf{Z}$ satisfy $a^2+2b^2 \equiv 0 \pmod 3$. Prove that either $a \equiv b \equiv 0 \pmod 3$ or neither a nor b is congruent to 0 modulo 3.

   Solution. We use indirect proof, i.e., prove the contrapositive. Suppose it is not true that

$$\text{(i)} \quad a \equiv b \equiv 0 \pmod 3, \quad \text{or} \quad \text{(ii)} \quad a \not\equiv 0 (\bmod 3) \text{ and } b \not\equiv 0 (\bmod 3).$$

   That is

   "one of the numbers a and b is divisible by 3, and the other is not divisible by 3."

   We may assume that without loss of generality that $a = 3k$ and $b = 3q \pm 1$ with $k, q \in \mathbf{Z}$. Then $a^2 + b^2 = 9k^2 + 9q^2 \pm 6q + 1 = 3(3k^2 + 3q^2 \pm 2q) + 1$, which is not divisible by 3.

8. Prove that if $n \in \mathbf{Z}$ is such that $n \equiv 3 \pmod 7$, then $n^2 \equiv 2 \pmod 7$.

   Solution. Suppose $n \equiv 3 \pmod 7$, i.e., $n = 7k+3$. Then

$$n^2 = (7k+3)^2 = 49k^2 + 42k + 9 = 7(7k^2+6k+1)+2$$

   so that $n^2 \equiv 2 \pmod 7$.