**Math 214 Foundations of Mathematics    Homework 4    Your name**

Five points for every problem.

1. For $\alpha \in (0,1)$, let $S_\alpha = (-\alpha, \alpha)$. Prove that

$$\cap_{\alpha\in(0,1)} S_\alpha = \{0\} \quad \text{and} \quad \cup_{\alpha\in(0,1)} S_\alpha = (-1,1).$$

   Hint: To prove $\cap_{\alpha\in(0,1)} S_\alpha = \{0\}$, first argue that $0 \in S_\alpha$ for every $\alpha \in (0,1)$ to conclude that $0 \in \cap_{\alpha\in(0,1)} S_\alpha$. Next, show that if $x \in \mathbf{R}$ and $x \neq 0$, then there is $\alpha \in (0,1)$ such that $x \notin S_\alpha$ to conclude that $x \notin \cap_{\alpha\in(0,1)} S_\alpha$.

2. Let $A, B$ be sets. Use the fact that $A - B = A \cap \overline{B}$ and the Distributive Law to prove

$$A = (A - B) \cup (A \cap B).$$

   [Hint: Use the fact that $A - B = A \cap \overline{B}$, and the distributive law.]

3. For any three sets $A, B, C$, show that $(A \times C) - (B \times C) = (A - B) \times C$.

4. Prove that if $a, b, c, d \in \mathbf{R}$, then $(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$.

5. Prove that if $x, y, z \in \mathbf{R}$, then $|x - z| \leq |x - y| + |y - z|$.

   Hint: Let $a = x - y$ and $b = y - z$. Consider the four cases according to $a$ and $b$ are positive or non-positive.

   **Recall** If two integers $x, y$ have the same remainder when they are divided by the positive integer $n \geq 2$, we write $x \equiv y \pmod{n}$.

6. Let $n \in \mathbf{Z}$. Prove that

$$2n^2 + 1 \equiv 0 \pmod{3} \quad \text{if and only if} \quad n \not\equiv 0 \pmod{3},$$

   i.e., $3 | (2n^2 + 1)$ if and only if $3 \not| n$.

7. Let $a, b \in \mathbf{Z}$ satisfy $a^2 + 2b^2 \equiv 0 \pmod{3}$. Prove that

   (i) $a \equiv b \equiv 0 \pmod{3}$, or (ii) $a \not\equiv 0 \pmod{3}$ and $b \not\equiv 0 \pmod{3}$.

8. Prove that if $n \in \mathbf{Z}$ is such that $n \equiv 3 \pmod{7}$, then $n^2 \equiv 2 \pmod{7}$.