Math 214 – Foundations of Mathematics **Homework 5** **Sample solution**

1. For each of the following sets, determine whether it is well-ordered and show your reasons.

   (a) $S = \{n \in \mathbf{N} : n \text{ is even}\}$. (b) $T = \{x \in \mathbf{Q} : x \geq 0\}$.

   Solution. (a) S is well-ordered. Any nonempty subset of S is a subset of $\mathbf{N}$ and hence has a minimum element because $\mathbf{N}$ is well-ordered.

   (b) T is not well-ordered because $T_1 = \{x \in \mathbf{Q} : x > 0\}$ is a subset of T that has no smallest element. To see this, note that $1 \in T_1$ so that T_1 is nonempty; for any $x \in T_1$ we have $y = x/2 \in T_1$ so that x cannot be a least element.

2. Prove that $\sum_{k=0}^{n}(2k+1) = (n+1)^2$ for all $n \in \mathbf{N}$.

   Proof. LHS of $P(1)$ is $1 + (2+1) = 4 = (1+1)^2$, which is the RHS of $P(1)$. So, $P(1)$ holds.

   Assume that $P(m)$ holds for $m \geq 1$, i.e., $\sum_{k=0}^{m}(2k+1) = (m+1)^2$.

   Consider $P(m+1)$. We have

$$\begin{aligned}\sum_{k=0}^{m+1}(2k+1) &= \sum_{k=0}^{m}(2k+1) + [2(m+1)+1] \\ &= (m+1)^2 + 2(m+1) + 1 = ((m+1)+1)^2 = (m+2)^2.\end{aligned}$$

   So, $P(m+1)$ holds. By the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbf{N}$.

3. Prove that $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \frac{n}{3n+9}$ for every positive integer n.

   Proof. When $n = 1$, we have $\frac{1}{(1+2)(1+3)} = \frac{1}{12} = \frac{1}{3 \cdot 1 + 9}$. So, $P(1)$ holds.

   Suppose $P(m)$ holds, i.e., $\sum_{k=1}^{m} \frac{1}{(k+2)(k+3)} = \frac{m}{3m+9}$. Consider $P(m+1)$. We have

$$\sum_{k=1}^{m+1} \frac{1}{(k+2)(k+3)} = \frac{m}{3m+9} + \frac{1}{(m+3)(m+4)} \qquad \text{by induction assumption}$$

$$= \frac{1}{m+3}\left(\frac{m}{3} + \frac{1}{m+4}\right) = \frac{1}{m+3}\frac{m(m+4)+3}{3(m+4)}$$

$$= \frac{1}{m+3}\frac{m^4+4m+3}{3(m+4)} = \frac{1}{m+3}\frac{(m+3)(m+1)}{3(m+4)} = \frac{m+1}{3(m+1)+9}$$

   So, $P(m+1)$ is true. By the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbf{N}$.

4. Determine (with proof) the set of integers n such that $n \geq 3$, $n^3 \leq 3^n$.

   Proof. When $n = 3$, $n^3 = 3^3 = 3^n$. So, $P(3)$ holds.

   Assume that $P(k)$ holds for $k \geq 3$, $k^3 \leq 3^k$ for some $k \geq 3$. Consider $P(k+1)$. We have

$$\begin{aligned}3^{k+1} &= 3 \cdot 3^k \geq 3 \cdot (k^3) && \text{by induction assumption} \\ &= k^3 + k^3 + k^3 \geq k^3 + 3k^2 + 3^2 k && \text{because } k \geq 3 \\ &\geq k^3 + 3k^2 + 3k + 1 = (k+1)^3. && \text{because } 9k \geq 3k+1\end{aligned}$$

   So, $P(k+1)$ holds. By the principle of mathematical induction, $n^3 \leq 3^n$ for all $n \geq 3$.

5. Prove $P(n) : 1 + \frac{1}{4} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all $n \in \mathbf{N}$.

   Proof. When $n = 1$, we have $1 \leq 1 - 1/1$. So, $P(1)$ holds.

   Suppose $P(k)$ holds, i.e., $1+\frac{1}{4}+\cdots+\frac{1}{k^2} \leq 2-\frac{1}{k}$. Consider $P(k+1)$. By induction assumption,

$$1 + \frac{1}{4} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

   because

$$2 - \frac{1}{k+1} - (2 - \frac{1}{k} + \frac{1}{(k+1)^2}) = \frac{(k+1)^2 - k(k+1) - k}{k(k+1)^2} = \frac{1}{k(k+1)^2} > 0.$$

   Thus, $P(k+1)$ holds. By the principle of mathematical induction, the results follows.

6. Prove $P(0) : 7|(3^{2n} - 2^n)$ for all $n \in \mathbf{N}$.

   Proof. We prove the result for all $n \in \mathbf{N} \cup \{0\}$. When $n = 0$, we see that $3^0 - 2^0 = 0$ is divisible by 7. So, $P(0)$ holds.

   Suppose $P(m)$ holds for $m \geq 0$. That is, $3^{2m} - 2^m = 7r$ for some $r \in \mathbf{Z}$. Consider $P(m+1)$. Then

$$3^{2(m+1)} - 2^{m+1} = 3^{2m}3^2 - 2^m 2 = 3^m 7 + 2(3^m - 2^m) = 7(3^m + 2r)$$

   is a multiple of 7. Thus, $P(m+1)$ holds.

   By the principle of mathematical induction, $P(n)$ holds for all $n \in \mathbf{N} \cup \{0\}$.

7. Prove that $12|(n^4 - n^2)$ for all $n \in \mathbf{N} \cup \{0\}$.

   Solution. If $n = 0$, we have $12|0$ so that the statement is true.

   Suppose $12|(m^4 - m^2)$ for an integer $m \geq 0$. Then $m^4 - m^2 = 12q$ for some $q \in \mathbf{Z}$. Note that

$$(m+1)^4 - (m+1)^2 = m^4 + 4m^3 + 6m^2 + 4m + 1 - m^2 - 2m - 1$$

$$= m^4 - m^2 + (4m^3 + 6m^2 + 2m) = 12q + 2m(m+1)(m+2).$$

   Consider three cases,

   (a) $m = 3k$: $m(m+1)(m+2) = 3k(m+1)(m+2) = 6kL_1$ as $(m+1)(m+2) = 2L_1$ is even;

   (b) $m = 3k+1$: $m(m+1)(m+2) = m(m+1)3(k+1) = 3(k+1)m(m+1) = 6L_2$ as $(m+1)(m+2) = 2L_2$ is even;

   (c) $m = 3k+2$: $m(m+1)(m+2) = m3(k+1)(m+2) = 6L_3m(m+2)$ if $k+1 = 2L_3$ is even, or $m(m+1)(m+2) = 6L_4(k+1)(m+2)$ with $m = 2L_4$ is even if k is even.

   So, in all cases, $m(m+1)(m+2)$ is divisible by 6. Thus, $2m(m+1)(+2)$ is divisible by 12, and $(m+1)^4 - (m+1)^2$ is divisible by 12. By the principle of mathematical induction, $12|(n^4 - n^2)$ for all nonnegative integer n.

   Remark It is relatively easy to show that $3|(m^4 - m^2)$ and $4|(m^4 - m^2)$. One may then conclude that $12|(m^4 - m^2)$. However, to get this conclusion, we need a result saying that if a and b have no common factor such that $a|n$ and $b|n$, then $ab|n$. This result has not been proved rigorously so that our proof avoid that.

8. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1, a_2 = 4, a_3 = 9$, and

$$a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n - 3)$$

for $n \geq 4$. Conjecture a formula for a_n and prove that your conjecture is correct.

Proof. Examining the first 4 or 5 terms, we conjecture that $P(n) : a_n = n^2$ for $n \in \mathbf{N}$.

Clearly, $P(1), P(2), P(3)$ hold.

Suppose $P(k)$ holds for $k = 1, \ldots, m$ for $m \geq 3$. **Note that we need to ensure the first 3 cases hold to prove the next case.** Consider $P(m + 1)$. We have

$$\begin{aligned} a_{m+1} &= a_m - a_{m-1} + a_{m-2} + 2(2(m+1) - 3) \\ &= m^2 - (m-1)^2 + (m-2)^2 + 2(2m-1) \quad \text{by induction assumption} \\ &= m^2 - (m^2 - 2m + 1) + (m^2 - 4m + 4) + 2(2m - 1) \\ &= m^2 + 2m + 1 = (m+1)^2. \end{aligned}$$

Thus, $P(m + 1)$ holds. By the general principle of mathematical induction, $P(n)$ holds for all $n \in \mathbf{N}$.

9. Show that an positive integer is a multiple of 9 if and only if the sum of all digits of the integer is a multiple of 9.

Proof. Suppose $x = a_m \cdots a_1 a_0 = a_m 10^m + \cdots + a_1 10^1 + a_0$, where $a_0, \ldots, a_m \in \{0, 1 \ldots, 9\}$ such that $a_m \neq 0$. Note that for $k \in \mathbf{N}$, we have $10^k = 1 + 9 \cdot (\underbrace{1 \cdots 1}_{n}) \equiv a_k \pmod 9$.

Thus, $x = a_m 10^m + \cdots + a_1 10^1 + a_0 \equiv a_m + \cdots + a_0 \pmod 9$.

Remark Here we may or may use induction to prove $10^n \equiv 1 \pmod 9$.