Math 214 Foundations of Mathematics Homework 5 **Your Name**

Five points for each problem.

1. For each of the following sets, determine whether it is well-ordered and show your reasons.

   (a) $S = \{n \in \mathbf{N} : n \text{ is even}\}$. (b) $T = \{x \in \mathbf{Q} : x \geq 0\}$.

2. Prove that $\sum_{k=0}^{n}(2k+1) = (n+1)^2$ for all $n \in \mathbf{N}$.

3. Prove that $\sum_{k=1}^{n} \frac{1}{(k+2)(k+3)} = \frac{n}{3n+9}$ for every positive integer n.

4. Determine (with proof) the set of integers n such that $n \geq 3$, $n^3 \leq 3^n$.

5. Prove that $1 + \frac{1}{4} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$ for all $n \in \mathbf{N}$.

6. Prove that $7 | (3^{2n} - 2^n)$ for all $n \in \mathbf{N}$.

7. Prove that $12 | (n^4 - n^2)$ for all $n \in \mathbf{N} \cup \{0\}$.

8. A sequence $\{a_n\}$ is defined recursively by $a_1 = 1, a_2 = 4, a_3 = 9$, and

   $$a_n = a_{n-1} - a_{n-2} + a_{n-3} + 2(2n-3)$$

   for $n \geq 4$. Conjecture a formula for a_n and prove that your conjecture is correct.

9. (Optional) Show that an positive integer is a multiple of 9 if and only if the sum of all digits of the integer is a multiple of 9.

   [Hint: Prove that $10^n \equiv 1 (\bmod 9)$.]