Math 214 Foundations of Mathematics   Homework 6   Your name

Solve the following problems. Five points each.

1. Let $n \in \mathbf{N}$ with $n > 1$. Suppose $a \equiv r \pmod{n}$ and $b \equiv s \pmod{n}$. Show that

$$a + b \equiv r + s \pmod{n} \text{ and } ab \equiv rs \pmod{n}.$$

   Recall that $\mathbf{Z}_n = \{[0], \ldots, [n-1]\}$ such that $[r] = \{nx + r : r \in \mathbf{Z}\}$. The above results show that if $[a] = [r]$ and $[b] = [s]$, then $[a + b] = [r + s]$ and $[ab] = [rs]$.

2. Use the result in the previous problem or otherwise to show that:

$$\text{(a) } 3^{2016} \equiv 1 (\operatorname{mod} 10). \qquad \text{(b) } 3^{2016} + 5^{2016} \equiv 2 (\operatorname{mod} 28).$$

3. Find $\gcd(51, 288)$ and $m, n \in \mathbf{Z}$ such that $\gcd(51, 288) = 51n + 288m$ using the Euclidean Algorithm.

4. Let a, b, c be integers. Prove that if $3|(abc - 1)$, then $3|(a - 1)$, $3|(b - 1)$, or $3|(c - 1)$.

5. Let $d = \gcd(a, b)$. If $a = da'$ and $b = db'$, show that $\gcd(a', b') = 1$.

6. (a) Find a pair of integers (x, y) such that $3x+2y = 1$, and show that $(x_n, y_n) = (x+2n, y-3n)$ also satisfies $3x_n + 2y_n = 1$ for every $n \in \mathbf{Z}$.

   (b) Let $a, b \in \mathbf{Z}$ such that $ab \neq 0$. Show that there are infinitely many pairs x, y of integers such that $\gcd(a, b) = ax + by$.

7. Show that $n + 1$ and $3n + 2$ are coprime.

8. Use the Fundamental Theorem of Arithmetic to prove that $\sqrt[3]{3}$ and $\log_{10} 234$ are irrational numbers.

   [Hint: If $\sqrt[3]{3} = p/q$, then $3q^3 = p^3$. Then ....]

9. (Extra credits) For integers a and b, let $\operatorname{lcm}(a, b)$ be the least positive multiplier of a and b.

   (a) Express $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$ in terms of prime factors of a and b.

   (b) Show that $\operatorname{lcm}(a, b) \cdot \gcd(a, b) = ab$.

   [Hint: Let $p_1, \ldots, p_k$ be the list of all prime factors of a and b. Then $a = p_1^{r_1} \cdots p_k^{r_k}$ and $b = p_1^{s_1} \ldots p_k^{s_k}$. Express (with proofs) the gcd and lcm of a and b in terms of $p_1, \ldots, p_k$. Then ...]