Math 214 – Foundations of Mathematics Homework 7 Sample Solution

1. (6 points) Solve the following problems in $\mathbf{Z}_n$.

   (a) In $\mathbf{Z}_8$, express the following sums and products as $[r]$, where $0 \le r < 8$:

   $$[3] + [6], \quad [3][6], \quad [-13] + [138], \quad [-13][138].$$

   (b) Let $[a], [b] \in \mathbf{Z}_8$. If $[a][b] = [0]$, does it follow that $[a] = [0]$ or $[b] = [0]$

   (c) Prove that for any prime p, if $[a], [b] \in \mathbf{Z}_p$, then $[a][b] = [0]$ implies $[a] = [0]$ or $[b] = [0]$.

   Solution. (a) In $\mathbf{Z}_8$, $[3] + [6] = [9] = [1]$, $[3][6] = [18] = [2]$, $[-13] + [138] = [3] + [2] = [5]$, and $[-13][138] = [3][2] = [6]$.

   (b) No. Let $a = 2$ and $b = 4$. Then $[a] \neq [0]$ and $[b] \neq 0$. But $[ab] = [8] = [0]$.

   (c) Suppose $[0] = [a][b] = [ab]$. Then $p|(ab)$. By Euclid's lemma, $p|a$ or $p|b$. (Alternatively, by the Fundamental Theorem of Arithmetic, p is a prime fact of ab so that p will appear in the prime number factorization of a or that of b.) Thus, $[a] = [0]$ or $[b] = [0]$.

2. (4 points) A relation R is defined on $\mathbf{Z}$ by $(a, b) \in R$ if $|a - b| \le 2$. Which of the properties reflexive, symmetric, and transitive does the relation R possess? Justify your answers.

   It is reflexive because for all $x \in \mathbf{Z}$, we have $|x - x| = 0 \le 2$ so that $(x, x) \in R$.

   It is symmetric because for any $x, y \in \mathbf{Z}$ satisfying $(x, y) \in R$ so that $|x - y| \le 2$, we have $|y - x| \le 2$ so that $(y, x) \in R$..

   It is not transitive because for $(x, y, z) = (0, 2, 4)$, we have $(x, y), (y, z) \in R$ as $|x - y| = |y - z| = 2$, but x is not related to z as $|x - z| = 4 > 2$.

3. (4 points) Let R be a relation defined on $\mathbf{Z} - \{0\}$ by $(a, b) \in R$ if $ab > 0$. Show that R is an equivalence relation on $\mathbf{Z} - \{0\}$.

   It is reflexive because for any $x \in \mathbf{Z} - \{0\}$, we have $x^2 > 0$ so that $(x, x) \in R$.

   It is symmetric because for any $x, y \in \mathbf{Z} - \{0\}$ such that $(x, y) \in R$, we have $xy > 0$, and hence $yx > 0$ so that $(y, x) \in R$.

   It is transitive because for any $x, y, z \in \mathbf{Z} - \{0\}$, if $(x, y), (y, z) \in R$, then $xy > 0$ and $yz > 0$. Hence $xyyz > 0$. Since $y^2 > 0$, we see that $xz > 0$ so that $(x, z) \in R$.

   Combining, we see that R is an equivalence relation.

4. (8 points) Find relations on $S = \{1, 2, 3\}$ satisfying the following. Verify your answers.

   (a) Reflexive, symmetric, not transitive.

   Answer: $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$.

   Reflexive: $(1, 1), (2, 2), (3, 3) \in R$.

   Symmetric: Direct checking shows if $(x, y) \in R$, then so is (y, x).

   Not Transitive: $(1, 2), (2, 3) \in \mathbf{R}$, but $(1, 3) \notin \mathbf{R}$.

   (b) Reflexive, not symmetric, not transitive.

   Answer: $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$.

Reflexive: $(1,1),(2,2),(3,3) \in R$.

Not Symmetric: $(1,2) \in R$, but $(2,1) \notin R$.

Not Transitive: $(1,2),(2,3) \in \mathbf{R}$, but $(1,3) \notin \mathbf{R}$.

(c) Symmetric, transitive, not reflexive.

Answer: $R = \{(1,1)\}$.

Easy to check using arguments in (a), (b).

(d) Symmetric, not reflexive, not transitive.

Answer: $R = \{(1,2),(2,1),(2,3),(3,2)\}$.

Easy to check using arguments in (a), (b).

5. (8 points) Find relations on $\mathbf{Z}$ satisfying the following. Verify your answers.

(a) Reflexive, symmetric, not transitive.

Answer: $R = \{(a,a) : a \in \mathbf{Z}\} \cup \{(1,2),(2,1),(2,3),(3,2)\}$.

Reflexive: For every $a \in \mathbf{Z}$, $(a,a) \in R$.

Symmetric: Direct checking shows if $(x,y) \in R$, then so is (y,x).

Not Transitive: $(1,2),(2,3) \in \mathbf{R}$, but $(1,3) \notin \mathbf{R}$.

(b) Reflexive, not symmetric, not transitive.

Answer: $R = \{(a,a) : a \in \mathbf{Z}\} \cup \{(1,2),(2,3)\}$.

Reflexive: For every $a \in \mathbf{Z}$, $(a,a) \in R$.

Not Symmetric: $(1,2) \in R$, but $(2,1) \notin R$.

Not Transitive: $(1,2),(2,3) \in \mathbf{R}$, but $(1,3) \notin \mathbf{R}$.

(c) Symmetric, transitive, not reflexive.

Answer: $R = \{(1,1)\}$.

Easy to check using arguments in (a), (b).

(d) Symmetric, not reflexive, not transitive.

Answer: $R = \{(1,2),(2,1),(2,3),(3,2)\}$.

Easy to check using arguments in (a), (b).

6. (a) (3 points) Define the relation R on $\mathbf{R}^2$ by $(x_1,y_1)R(x_2,y_2)$ if $|x_1| + |y_1| = |x_2| + |y_2|$. Prove that R is an equivalence relation, and describe the geometrical shape of the disjoint equivalence classes of R in $\mathbf{R}^2$.

(b) (3 points) Consider the partition of $\mathbf{R}^2$ by straight lines $L_r = \{(x,y) : x+y = r\}$ for each $r \in \mathbf{R}$. Show that $P = \{L_r : r \in \mathbf{R}\}$ is a partition of $\mathbf{R}^2$, and define an equivalence relation so that L_r's are the equivalence classes.

Solution. (a) Reflexive: Let $(x,y) \in \mathbf{R}^2$. Then $|x| + |y| = |x| + |y|$. So, $(x,y)R(x,y)$.

Symmetric: If $(x_1,y_1)R(x_2,y_2)$, then $|x_1| + |y_1| = |x_2| + |y_2|$. So, $|x_2| + |y_2| = |x_1| + |y_1|$ and hence $(x_2,y_2)R(x_1,y_1)$.

Transitive: If $(x_1,y_1)R(x_2,y_2)$ and $(x_2,y_2)R(x_3,y_3)$, then $|x_1|+|y_1| = |x_2|+|y_2| = |x_3|+|y_3|$. So, $(x_1,y_1)R(x_3,y_3)$, The equivalence classes are $[(r,0)] = \{(x,y) : r = |x| + |y|\}$, which equals to the square with vertexes $(r,0),(0,r),(-r,0),(0,-r)$, for each $r \geq 0$.

(b) We prove that $\{L_r : r \in \mathbf{R}\}$ is a partition of $\mathbf{R}^2$ in the following.

First, $(r,0) \in L_r$ is non-empty for every $r \in \mathbf{R}$.

Second, if $L_r \neq L_s$, i.e., $r \neq s$, then $L_r \cap L_s = \emptyset$. Otherwise, there is $(x,y) \in L_r \cap L_s$ so that $x + y = r$ and $x + y = s$, which is impossible as $r \neq s$.

Third, we have $\mathbf{R}^2 = \cup_{r\in\mathbf{R}} L_r$. Clearly, $L_r \subseteq \mathbf{R}^2$ for every $r \in \mathbf{R}$ so that $\cup_{r\in\mathbf{R}} L_r \subseteq \mathbf{R}^2$. If $(x,y) \in \mathbf{R}^2$, then $(x,y) \in L_s$ with $s = x + y$. Thus, $(x,y) \in \cup_{r\in\mathbf{R}} L_r$. Thus, $\mathbf{R}^2 \subseteq \cup_{r\in\mathbf{R}} L_r$. Combining, we see that $\mathbf{R}^2 = \cup_{r\in\mathbf{R}} L_r$.

By the theorem proved in class, we can define $(x_1,y_1)R(x_2,y_2)$ if there is $r \in \mathbf{R}$ such that $(x_1,y_1),(x_2,y_2) \in L_r$. Equivalently, there is $r \in \mathbf{R}$ such that $x_1 + y_1 = r = x_2 + y_2$. One can simply define $(x_1,y_1)R(x_2,y_2)$ if $x_1 + y_1 = x_2 + y_2$.

7. Determine with explanation all the equivalence relation on $\{1,2,3\}$.

   Solution. There are five equivalence relations on $\{1,2,3\}$ corresponding to the following 5 partitions of $\{1,2,3\}$.

   (a) Partition $P = \{\{1\},\{2\},\{3\}\}$, relation $R = \{(1,1),(2,2),(3,3)\}$.

   (b) Partition $P = \{\{1\},\{2,3\}\}$, relation $R = \{(1,1),(2,2),(3,3),(2,3),(3,2)\}$.

   (c) Partition $P = \{\{2\},\{1,3\}\}$, relation $R = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$.

   (d) Partition $P = \{\{3\},\{1,2\}\}$, relation $R = \{(1,1),(2,2),(3,3),(1,2),(2,1)\}$.

   (d) Partition $P = \{\{1,2,3\}\}$, relation $R = \{(i,j) : i,j = 1,2,3\}$.

   By the theorems shown in class (textbook), there is a one-one correspondence between partitions and equivalence relations on $A = \{1,2,3\}$. Clearly, there are only five partitions for A. So, there are five equivalence relations described above.