Math 214 Foundations of Mathematics Homework 8 Sample solution

1. Determine the largest sets $A, B \subseteq \mathbf{R}$ such that $f : A \to B$ defined by $f(x) = \sqrt{3x-1}$ is a function. Determine whether the resulting function is injective.

   Solution. To ensure that every $x \in A$, we get $f(x) \in \mathbf{R}$, we require $3x - 1 \geq 0$. So, $A = \{x \in \mathbf{R} : x \geq 1/3\} = [1/3, \infty)$.

   The set $B = \mathbf{R}$ is the largest possible subset.

   Now, suppose $f(x) = f(y)$. Then $\sqrt{3x-1} = \sqrt{3y-1}$. So, $3x - 1 = 3y - 1$, i.e., $x = y$. Hence, f is injective.

2. Consider $h : \mathbf{Z}_{16} \to \mathbf{Z}_{24}$ by $h([a]_{16}) = [3a]_{24}$ for each $a \in \mathbf{Z}$.

   (a) Prove that h is a function.

   (b) Is h injective? surjective? bijective?

   Solution. (a) Clearly, for every $[a]_{16} \in \mathbf{Z}_{16}$, $f([a]_{16}) = [3]_{24} \in \mathbf{Z}_{24}$. Furthermore, if $[a]_{16} = [b]_{16}$, then $a - b = 16q$ for some $q \in \mathbf{Z}$. So, $3a - 3b = 48q = 24(2q)$. Thus, $h([a]_{16}) = [3a]_{24} = [3b]_{24} = h([b]_{16})$. Thus, h is a well defined function.

   (b) In $\mathbf{Z}_{16}$, we have $[0]_{16} \neq [8]_{16}$, but $h([0]_{16}) = [0]_{24} = [24]_{24} = h([8]_{16})$. So, h is not injective.

   Because $\mathbf{Z}_{16}$ has only 16 elements. $h(\mathbf{Z}_{16})$ will have at most 16 elements, where as $\mathbf{Z}_{24}$ has 24 elements. So, h is not surjective.

   By the above, we see that f is not bijective.

3. Let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ where, for $(a, b) \in \mathbf{R} \times \mathbf{R}$, $f(a, b) = (2a + 7, 3b - 3)$. Prove that f is a bijective function and find f^{-1}.

   Solution. Injective. If $f(a_1, b_1) = f(a_2, b_2)$, then $(2a_1 + 7, 3b_1 - 3) = (2a_2 + 7, 3b_2 - 3)$. So, $2a_1 + 7 = 2a_2 + 7$ and $3b_1 - 3 = 3b_2 - 3$. Thus, $(a_1, b_1) = (a_2, b_2)$.

   Surjective. Suppose $(x, y) \in \mathbf{R}^2$. We want to choose (a, b) such that $(x, y) = f(a, b) = (2a + 7, 3b - 3)$, i.e., $x = 2a + 7, y = 3b - 3$. Thus, $(a, b) = ((x - 7)/2, (y + 3)/3)$ satisfies the required condition.

   By the above proof, we see that f is bijective.

   In the proof of on-to property, we see that $f^{-1} : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$, $f^{-1}(x, y) = ((x - 7)/2, (y + 3)/3)$.

4. Define $f : \mathbf{N} \to \mathbf{Z}$ by $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (1-n)/2 & \text{if } n \text{ is odd.} \end{cases}$

   Show that f is a well-defined bijection.

   Solution. Well-defined property. If $a = b$ is even, then $f(a) = f(b) = a/2$. If $a = b$ is odd, then $f(a) = f(b) = (1 - a)/2$. So, f is well defined.

   Note that f maps even numbers to positive integers, and odd numbers to non-positive integers.

   One-one. Suppose $f(a) = f(b)$. We consider two cases. If $f(a) = f(b) > 0$, then a and b must be even so that $a/2 = f(a) = f(b) = b/2$ so that $a = b$. If $f(a) = f(b) \leq 0$, then a and b must be odd so that $(1 - a)/2 = f(a) = f(b) = (1 - b)/2$ so that $a = b$.

   Onto. Suppose $z \in \mathbf{Z}$. If $z > 0$, then $f(2z) = z$. If $z \leq 0$, then $a = 1 - 2z$ satisfies $f(a) = z$.

5. Suppose A is a non-empty set. Determine the functions $f : A \to A$ that are also equivalence relations.

   Solution. Suppose $f : A \to A$ is a function and also an equivalence relation. Then for any $a \in A$, $(a, a) \in f$ so that $f(a) = a$. Now, it is easy to check that $f = \{(a, a) : a \in A\}$ is an equivalence relation. So, the only function $f : A \to A$ that is also an equivalence relation if the identity function i_A.

6. Let A, B and C be nonempty sets and let f, g and h be functions such that $f : A \to B, g : B \to C$ and $h : B \to C$. For each of the following, prove or disprove:

(a) if $g \circ f = h \circ f$, then $g = h$.

(b) if f is surjective and $g \circ f = h \circ f$, then $g = h$.

Solution. (a) The statement is false. Here is a counter-example for both parts. Let $A = \{1\}$, $B = \{a, b\}$, $C = \{a, b\}$. Suppose $f(1) = a$, $g(a) = g(b) = a$ and $h(a) = a, h(b) = b$. Then $g \circ f = h \circ f$, but $g \neq h$.

(b) For any $b \in B$ there is $a \in f$ such that $f(a) = b$. Thus, $g(b) = g(f(a)) = h(f(a)) = h(b)$.

7. For nonempty sets A, B, C, let $f : A \to B$ and $g : B \to C$ be functions.

(a) Prove that if $g \circ f$ is injective, then f is injective.

(b) Disprove that if $g \circ f$ is injective, then g is injective.

Solution. (a) If $a_1 \neq a_2 \in A$ and $f(a_1) = f(a_2) = b \in B$. Let $g(b) = c$. Then $gf(a_1) = gf(a_2)$ so that $g \circ f$ is not injective.

(b) Let $A = B = C = \mathbf{Z}$, $f(x) = 2x$ and $g(x) = x/2$ if x is even, and $g(x) = x$ if x is odd. Then $g \circ f(x) = x$ for all x is injective, but g is not injective, say, $g(1) = g(2) = 1$.

8. For nonempty sets A and B and functions $f : A \to B$ and $g : B \to A$ suppose that $g \circ f = i_A$, the identity function on A.

(a) (4 points) Show that f is injective and g is surjective.

(b) (2 points) Show that f is not necessarily surjective.

(c) (2 points) Show that g is not necessarily injective.

Solution. (a) If f is not injective, then there are $a_1 \neq a_2$ in A such that $f(a_1) = f(a_2)$ in B. The $g \circ f(a_1) = g \circ f(a_2)$ so that $g \circ f$ is not the identity function.

If g is not surjective, then there is $a \in A$ such that $g(b) \neq a$ for any $b \in B$. Thus, $g(f(a)) = g(b) \neq a$ so that $g \circ f$ is not the identity function.

For (b) and (c). Let $A = \{1\}$, $B = \{a, b\}$, $f(1) = a$, $g(a) = g(b) = 1$. Then $g \circ f = i_A$, f is not injective, and g is not injective.

9. (6 points) Let $f : A \to B$ be a function, and let $A_1, A_2 \subseteq A$.

(a) Prove that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

(b) Prove that if f is injective, then $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$.

(c) Give an example showing that $f(A_1) \cap f(A_2) \not\subseteq f(A_1 \cap A_2)$ if f is not injective.

Proof. (a) If $b \in f(A_1 \cap A_2)$, then there is $a \in A_1 \cap A_2$ such that $f(a) = b$. Because $a \in A_1 \cap A_2$, $b = f(a) \in f(A_1)$ and $b = f(a) \in f(A_2)$. Thus, $b = f(a) \in f(A_1) \cap f(A_2)$.

(b) If f is injective, and $b \in f(A_1) \cap f(A_2)$, then $b = f(a_1)$ with $a_1 \in A_1$ and $b = f(a_2)$ with $a_2 \in A_2$. Because f is injective, $a_1 = a_2 = a \in A_1 \cap A_2$. Thus, $b = f(a) \in f(A_1 \cap A_2)$.

(c) Let $A = \{1, 2\}$, $B = \{b\}$, and $f(1) = f(2) = a$. For $A_1 = \{1\}, A_2 = \{2\}$, we have $f(A_1) \cap f(A_2) = \{b\}$, but $f(A_1 \cap A_2) = f(\emptyset) = \emptyset$.