Math 214 Foundations of Mathematics Homework 8 Your name

Five points for each problem unless specified otherwise.

1. Determine the largest sets $A, B \subseteq \mathbf{R}$ such that $f : A \to B$ defined by $f(x) = \sqrt{3x-1}$ is a function. Determine whether the resulting function is injective.

2. Consider $h : \mathbf{Z}_{16} \to \mathbf{Z}_{24}$ by $h([a]) = [3a]$ for each $a \in \mathbf{Z}$.

   (a) Prove that h is a function.

   (b) Is h injective? surjective? bijective?

   [Note: In (a), one has to show that if $[a] = [b]$ in $\mathbf{Z}_{16}$, then $f([a]) = f([b])$ in $\mathbf{Z}_{24}$.]

3. Let $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times \mathbf{R}$ where, for $(a, b) \in \mathbf{R} \times \mathbf{R}$, $f(a, b) = (2a + 7, 3b - 3)$. Prove that f is a bijective function and find f^{-1}.

   [Note: To prove f is well-defined, one has to show that every $(x, y) \in \mathbf{R}\times\mathbf{R}$, $f(x, y) = (u, v)$ is an element in $\mathbf{R}\times\mathbf{R}$. To prove that f is surjective, one has to show that for every $(u, v) \in R\times\mathbf{R}$ there is $(x, y) \in \mathbf{R} \times \mathbf{R}$ such that $f(x, y) = (u, v)$. In such a way, $f^{-1}(u, v) = (x, y)$.]

4. Define $f : \mathbf{N} \to \mathbf{Z}$ by $f(n) = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (1-n)/2 & \text{if } n \text{ is odd.} \end{cases}$

   Show that f is a well-defined bijection.

   [Note: To show that f is well-defined, one has to show that for every $n \in \mathbf{N}$ $f(n) \in \mathbf{Z}$. You may need to consider two cases: n is even, n is odd. To prove that f is injective, one has to show that $f(n_1) = f(n_2)$ then $n_1 = n_2$. You may need to consider four cases: n_1 is even or odd, n_2 is even or odd. To prove that f is surjective, one has to show that for every $z \in \mathbf{Z}$, one can find $n \in \mathbf{N}$ such that $f(n) = z$. You need to consider two cases: $z > 0$ and $z \leq 0$.]

5. Suppose A is a non-empty set. Determine the functions $f : A \to A$ that are also equivalence relations.

   [Hint: Try the cases when $A = \{1, 2\}$ and $A = \{1, 2, 3\}$, and determine the general result.]

6. Let A, B and C be nonempty sets and let f, g and h be functions such that $f : A \to B, g : B \to C$ and $h : B \to C$. For each of the following,prove or disprove:

   (a) if $g \circ f = h \circ f$, then $g = h$.

   (b) if f is surjective and $g \circ f = h \circ f$, then $g = h$.

   [Hint: For (a), consider $f : \{a\} \to \{a, b\}$, and $g : \{a, b\} \to \{a, b\}$.]

7. For nonempty sets A, B, C, let $f : A \to B$ and $g : B \to C$ be functions.

   (a) Prove that if $g \circ f$ is injective, then f is injective.

   (b) Disprove that if $g \circ f$ is injective, then g is injective.

8. For nonempty sets A and B and functions $f : A \to B$ and $g : B \to A$ suppose that $g \circ f = i_A$, the identity function on A.

   (a) (4 points) Show that f is injective and g is surjective.

   (b) (2 points) Show that f is not necessarily surjective.

   (c) (2 points) Show that g is not necessarily injective.

9. (Extra 6 points) Let $f : A \to B$ be a function, and let $f(X) = \{f(x) : x \in X\}$ for any $X \subseteq A$. Suppose $A_1, A_2 \subseteq A$.

   (a) Prove that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$.

   (b) Prove that if f is injective, then $f(A_1) \cap f(A_2) \subseteq f(A_1 \cap A_2)$.

   (c) Give an example showing that $f(A_1) \cap f(A_2) \not\subseteq f(A_1 \cap A_2)$ if f is not injective.

Relevant definitions and terminology

- A relation $f : A \to B$ is a well-defined function if for every $a \in A$ there is a unique $b \in B$ such that $(a, b) \in f$, and we write $b = f(a)$.
- Let $f : A \to B$ be a function.
- The function f is injective if for any $a, b \in A$ such that $a \neq b$, it follows that $f(a) \neq f(b)$ in B. The function is not injective if there are $a, b \in A$ such that $a \neq b$, but $f(a) = f(b)$ in B.
- The function f is surjective if for every $b \in B$ there is $a \in A$ such that $f(a) = b$. The function is not surjective if there is $b \in B$ such that $f(a) \neq b$ for any $a \in A$.
- The function f is bijective if f is injective and surjective.
- For any function $R \subseteq A \times B$, $R^{-1} = \{(b, a) : (a, b) \in R\}$. If $f : A \to$ is a bijective function , then $f^{-1} = \{(b, a) : (a, b) \in f\}$ is a function (bijection) and we write $b^{-1}(b) = a$.
- Suppose $X \subseteq A$. Then $f(X) = \{f(x) : x \in X\} \subseteq B$.
- Suppose $g : B \to C$ is a function. Then $g \circ f : A \to C$ is the composite function such that $g \circ f(a) = g(f(a))$.