Math 214 – Foundations of Mathematics   Homework 9   Sample Solution

1. Let $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ be functions. Show that $f : A_1 \times A_2 \to B_1 \times B_2$ defined by $f(a_1, a_2) = (f_1(a_1), f_2(a_2))$ for every pair $(a_1, a_2) \in A_1 \times A_2$, is a function.

   (a) Show that if f_1, f_2 are injective. Then f is injective.

   (b) Show that if f_1, f_2 are surjective. Then f is surjective.

   Solution. To show that f is well defined. Suppose $(a_1, a_2) \in A_1 \times A_2$. Then $a_1 \in A_1, a_2 \in A_2$. Hence there is a unique $b_1 \in B_1$ and a unique $b_2 \in B_2$ such that $f_1(a_1) = b_1 \in B_1, f_2(a_2) = b_2 \in B_2$. Hence, we get a unique pair $(b_1, b_2) \in B_1 \times B_2$ such that $f(a_1, a_2) = (b_1, b_2) \in B_1 \times B_2$.

   (a) Suppose f_1, f_2 are injective. Now, suppose $f((a_1, a_2)) = f((\alpha_1, \alpha_2))$. Then $(f_1(a_1), f_2(a_2)) = (f_1(\alpha_1), f_2(\alpha_2))$, i.e., $f_1(a_1) = f_1(\alpha_1)$ and $f_2(a_2) = f_2(\alpha_2)$. Thus, $a_1 = \alpha_1$ and $a_2 = \alpha_2$. Because f_1 and f_2 are injective, $(a_1, a_2) = (\alpha_1, \alpha_2)$. So, f is injective.

   (b) Assume f_1, f_2 are surjective. Suppose $(b_1, b_2) \in B_1 \times B_2$. Because f_1, f_2 are surjective, there exist $a_1 \in A_1$ and $a_2 \in A_2$ such that $f_1(a_1) = b_1$ and $f_2(a_2) = b_2$. Thus, $f(a_1, a_2) = (b_1, b_2)$. So, f is surjective.

2. (a) Let $f : A \to B$ be a function. Define a relation R on A by $(a_1, a_2) \in R$ if $f(a_1) = f(a_2)$. Show that R is an equivalence relation.

   (b) Suppose $S \subseteq A \times B$. Define a relation $\hat{R}$ on A by $(a_1, a_2) \in \hat{R}$ if there is $b \in B$ such that $(a_1, b), (a_2, b) \in S$. Prove or disprove that $\hat{R}$ is an equivalence relation.

   Solution. (a) Reflexive: Let $a \in A$. Then $f(a) = f(a)$. So, $(a, a) \in R$.

   Symmetric: If $(a, b) \in R$, then $f(a) = f(b)$. So, $f(b) = f(a)$ and $(b, a) \in R$.

   Transitive: If $(a, b), (b, c) \in R$, then $f(a) = f(b) = f(c)$. Thus $f(a) = f(c)$ so that $(a, c) \in R$.

   Combining the above, we see that R is an equivalence relation.

   (b) Let $A = \{1, 2\}$, $B = \{b\}$, and $S = \{(1, b)\}$. Then $(2, 2) \notin \hat{R}$. So, $\hat{R}$ is not reflexive. So, $\hat{R}$ is not an equivalence relation. [Actually, one can check that $\hat{R} = \{(1, 1)\}$.]

3. Construct $f : A \to B$ and $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g \neq i_B$, where i_A is the identity function on A and i_B is the identity function on B in each of the following cases.

   (a) $A = \{1\}$, $B = \{1, 2\}$.

   (b) $A = B = \mathbf{N}$.

   Solution. (a) $f(1) = 1$, $g(x) = 1$ for $x = 1, 2$. Then $g \circ f$ is the identity, but $f \circ g(1) = f \circ g(2) = 1$ so that $f \circ g$ is not the identity map.

   (b) Let $f(x) = 2x$, and $g(x) = \begin{cases} x/2 & \text{if } x \text{ is even,} \\ 1 & \text{otherwise.} \end{cases}$

   Then $g \circ f(x) = g(f(x)) = g(2x) = x$ for all $x \in \mathbf{N}$ so that $g \circ f = i_A$,

   but $f \circ g(x) = \begin{cases} x & \text{if } x \text{ is even,} \\ 2 & \text{otherwise.} \end{cases}$ Thus, $f \circ g$ is not the identity map.

4. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$ be permutations in S_5.

   Determine $\alpha \circ \beta$, $\beta \circ \alpha$, and β^{-1}.

   Solution.

$$\alpha \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 3 & 5 & 2 \end{pmatrix} \quad \beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 4 & 1 & 3 \end{pmatrix} \quad \beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{pmatrix}.$$

5. Let $A = \{2^a 3^b : a, b \in \mathbf{N}\}$. Show that $|\mathbf{N} \times \mathbf{N}| = |A|$.

   Solution. Define $f : \mathbf{N} \times \mathbf{N} \to A$ by $f((a,b)) = 2^a 3^b$. It is well defined because every pair $(a,b) \in \mathbf{N} \times \mathbf{N}$ generate a unique number $2^a 3^b \in A$.

   To prove that f is one-one, note that if $f((a,b)) = f((\alpha,\beta))$, then $2^a 3^b = 2^\alpha 3^\beta$. By the fundamental theorem of arithmetic, we see that $a = \alpha$ and $b = \beta$. Thus, $(a,b) = (\alpha,\beta)$.

   For any $y = 2^a 3^b \in A$, we have $(a,b) \in \mathbf{N} \times \mathbf{N}$ such that $f((a,b)) = y$. So, f is surjective.

6. Show that $f : \mathbf{R} \to (-1,1)$ defined by $f(x) = \frac{x}{1+|x|}$ is a bijection, i.e., $|(-1,1)| = |\mathbf{R}|$.

   Solution. It is well defined because every $x \in \mathbf{R}$, $f(x) = \frac{x}{1+|x|}$ is uniquely determined and $|f(x)| = |x|/(1+|x|) < 1$ so that $f(x) \in (-1,1)$.

   To prove that f is injective, suppose $f(x) = f(y)$. We need to prove that $x = y$.

   Case 1. If $f(x) = f(y) \geq 0$, then $x, y \geq 0$ so that $x/(1+x) = f(x) = f(y) = y/(1+y)$.

   Thus, $1/(1+x) = 1 - f(x) = 1 - f(y) = 1/(1+y)$. So, $1 + x = 1 + y$, i.e., $x = y$.

   Case 2. If $f(x) = f(y) < 0$, then $x, y < 0$ and $x/(1-x) = f(x) = f(y) = y/(1-y)$.

   Thus, $1/(1-x) = 1 + f(x) = 1 + f(y) = 1/(1-y)$. So, $1 - x = 1 - y$, i.e., $x = y$.

   Combining the two cases, we see that f is injective.

   To prove that f is surjective, suppose $y \in (-1,1)$.

   Case 1. If $y \in [0,1)$, we consider $x \geq 0$ such that $y = f(x) = x/(1+x)$, i.e., $y + xy = x$.

   So, we can take $x = y/(1-y) \in \mathbf{R}$ so that $f(x) = x/(1+x) = \frac{y}{1-y}/(1 + \frac{y}{1-y}) = y$.

   Case 2. If $y \in (-1,0)$, we consider $x < 0$ such that $y = f(x) = x/(1-x) \in \mathbf{R}$, i.e., $y - xy = x$.

   So, we can take $x = y/(1+y)$ so that $f(x) = x/(1-x) = \frac{y}{1+y}/(1 - \frac{y}{1+y}) = y$.

   Combining the two cases, we see that f is surjective.

7. (a) Construct a bijection from $f : \{0\} \cup \mathbf{N} \to \mathbf{N}$.

   (b) Construct a bijection from $g : \mathbf{Q} \to \mathbf{Q} - \{0\}$.

   Solution. Define $f : \{0\} \cup \mathbf{N} \to \mathbf{N}$ by $f(x) = x + 1$.

   Then for every $x \in \{0\} \cup \mathbf{N}$, $f(x) = x + 1 \in \mathbf{N}$ so that f is well defined.

   Clearly, $f(a) = f(b)$ implies $a + 1 = b + 1$ so that $a = b$. So, f is injective.

   If $y \in \mathbf{N}$, then $x = y - 1 \in \{0\} \cup \mathbf{N}$ satisfies $f(x) = x + 1 = y$. So, f is surjective.

   Thus, f is a well-defined bijection.

   (b) Define $g : \mathbf{Q} \to \mathbf{Q} - \{0\}$ by

$$g(x) = \begin{cases} x + 1 & \text{if } x \in \{0\} \cup \mathbf{N}, \\ x & \text{otherwise.} \end{cases}$$

   Suppose $x \in \mathbf{Q}$. If $x \in \{0\} \cup \mathbf{N}$, then $g(x) = x \in \mathbf{Q} - \{0\}$; if $x \notin \{0\} \cup \mathbf{N}$, then $g(x) = x \in \mathbf{Q} - \{0\}$. So, the function g is well-defined.

   Suppose $g(a) = g(b)$. If $g(a) = g(b) = y \in \mathbf{N}$, then we must have $a = y - 1 = b$; if $g(a) = g(b) = y \notin \mathbf{N}$, then $a = y = b$. So, g is injective.

   Suppose $y \in \mathbf{Q} - \{0\}$. If $y \in \mathbf{N}$, we can let $x = y - 1$ so that $g(x) = y$; if $y \notin \mathbf{N}$, we can let $x = y$ so that $g(x) = y$. So, g is surjective.

8. Show that $f : (0, 1] \to (0, 1)$ defined by

$$f(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \text{ with } n \in \mathbf{N}, \\ x & \text{otherwise,} \end{cases}$$

   is a bijection.

   Solution. For every $x \in (0, 1]$, if $x = 1/n$ for some $n \in \mathbf{N}$ then $f(x) = 1/(n + 1) \in (0, 1)$; if $x \neq 1/n$ for any $n \in N$ then $f(x) = x \in (0, 1)$. Thus, f is well-defined.

   Suppose $f(a) = f(b) = y \in (0, 1)$. If $y = 1/m$ for some $m \in \mathbf{N} - \{1\}$ then $a = 1/(m - 1) = b$. If $y \neq 1/m$ for any $m \in \mathbf{N}$ then $a = y = b$. Thus, f is injective.

   Suppose $y \in (0, 1)$. If $y = 1/m$ for some $m \in \mathbf{N} - \{1\}$, then for $x = 1/(m - 1)$ we have $f(x) = y$; if $y \neq 1/m$ for any $m \in \mathbf{N}$, then for $x = y$ we have $f(x) = f(y)$. So, f is surjective.

   Combining the above, we see that f is a bijection.