Math 214 Foundations of Mathematics   Homework 9   Your name

Five points for each question unless specified otherwise.

1. Let $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$ be functions. Show that $f : A_1 \times A_2 \to B_1 \times B_2$ defined by $f(a_1, a_2) = (f_1(a_1), f_2(a_2))$ for every pair $(a_1, a_2) \in A_1 \times A_2$, is a function.

   (a) Show that if f_1, f_2 are injective. Then f is injective.

   (b) Show that if f_1, f_2 are surjective. Then f is surjective.

2. (a) Let $f : A \to B$ be a function. Define a relation R on A by $(a_1, a_2) \in R$ if $f(a_1) = f(a_2)$. Show that R is an equivalence relation.

   (b) Suppose $S \subseteq A \times B$. Define a relation $\hat{R}$ on A by $(a_1, a_2) \in \hat{R}$ if there is $b \in B$ such that $(a_1, b), (a_2, b) \in S$. Prove or disprove that $\hat{R}$ is an equivalence relation.

3. Construct $f : A \to B$ and $g : B \to A$ such that $g \circ f = i_A$ and $f \circ g \neq i_B$, where i_A is the identity function on A and i_B is the identity function on B in each of the following cases.

   (a) $A = \{1\}$, $B = \{1, 2\}$.

   (b) $A = B = \mathbf{N}$.

4. Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 2 & 4 & 1 \end{pmatrix}$ be permutations in S_5.
   Determine $\alpha \circ \beta$, $\beta \circ \alpha$, and β^{-1}.

5. Let $A = \{2^a 3^b : a, b \in \mathbf{N}\}$. Construct a bijection from $\mathbf{N} \times \mathbf{N}$ to A.

   In such a case, we say that $\mathbf{N} \times \mathbf{N}$ and A has the same cardinality, denoted by $|\mathbf{N} \times \mathbf{N}| = |A|$.

6. Show that $f : \mathbf{R} \to (-1, 1)$ defined by $f(x) = \frac{x}{1+|x|}$ is a bijection, i.e., $|(-1, 1)| = |\mathbf{R}|$.

7. (a) Construct (with proof) a bijection from $f : \{0\} \cup \mathbf{N} \to \mathbf{N}$.

   (b) Construct (with proof) a bijection from $g : \mathbf{Q} \to \mathbf{Q} - \{0\}$.

   Hint for (b). Partition the domain into $A_1 \cup A_2$ with $A_1 = \{0, 1, 2, \ldots\}$, and partition the co-domain into $\mathbf{N} \cup A_2$. Construct $f : \mathbf{Q} \to \mathbf{Q} - \{0\}$ by $f(x) = \begin{cases} x + 1 & \text{if } x \in A_1, \\ x & \text{if } x \in A_2. \end{cases}$

8. (Extra credits) Show that $f : (0, 1] \to (0, 1)$ defined by

$$f(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n}, \\ x & \text{otherwise,} \end{cases}$$

   is a bijection.