1. Show that if A and B are denumerable sets, then $A \cup B$ is also denumerable

   Solution. Let $A = \{a_1, a_2, \ldots\}$, and let $C = B - A$. Then $A \cup B = A \cup C$. Because $C \subseteq B$. One of the following holds.

   Case 1. If $C = \emptyset$, then $A \cup C = A$ is denumerable.

   Case 2. If $C = \{c_1, \ldots, c_m\}$ is finite, then $f : A \cup C \to \mathbf{N}$ defined by $f(x) = \begin{cases} j & \text{if } x = c_j, \\ m + j & \text{if } x = a_j, \end{cases}$ is a bijection. Clearly, every $x \in A \cup C$ is sent to $\mathbf{N}$ by the definition. So, f is well defined. If $x_1 \neq x_2$ in $A \cup C$, then either $x_1, x_2 \in C, x_1, x_2, \in A$ or x_1, x_2 lie in different sets A and C. In all cases, we wee that $f(x_1) \neq f(x_2)$. Suppose $n \in \mathbf{N}$. If $n \leq m$, then $f(c_n) = n$; if $n > m$, then $f(a_{n-m}) = n$.

   Alternatively, one can define $f_1 : C \to \{1, \ldots, m\}$ by $f_1(c_j) = j$ and $f_2 : A \to \{m+1, m+2, \ldots\}$ by $f(a_j) = m + j$. Then f_1, f_2 are bijections, and therefore f is a bijection (by the divide and conquer lemma).

   Case 3. If $C = \{c_1, c_2, \ldots\}$ is denumerable, then $f : A \cup C \to \mathbf{N}$ defined by

   $f(x) = \begin{cases} 2j & \text{if } x = c_j, \\ 2j - 1 & \text{if } x = a_j, \end{cases}$ is a bijection. Proof is similar to Case 2.

   Remark Here we construct a bijection between $A \cup B$ and $\mathbf{N}$ to conclude $|A \cup B| = |\mathbf{N}|$. In Problem 5, we prove that the union of countable sets is countable by using a theorem discussed in class.

2. Prove that $S = \{(a, b) : a, b \in \mathbf{N}, a \geq 2b\}$ is denumerable.

   Solution. Let $f : S \to \mathbf{N}$ defined by $f(a, b) = 2^a 3^b$. Then $f(a, b) = f(\alpha, \beta)$ implies $2^a 3^b = 2^\alpha 3^\beta$ so that $a = \alpha$ and $b = \beta$. Thus, $(a, b) = (\alpha, \beta)$. So, f is a one-one function. We can conclude that $|S| = |f(S)|$ by the following.

   Lemma Suppose $g : A \to B$ is one-one. Then $|A| = |g(A)|$.

   Proof. Consider the function $\tilde{g} : A \to g(A)$ defined by $\tilde{g}(x) = g(x)$. Then $\tilde{g}(a) = \tilde{g}(b)$ implies that $g(a) = g(b)$, and hence $a = b$. So, $\tilde{g}$ is one-one. For any $y \in g(A)$, there is $x \in A$ such that $y = g(x) = \tilde{g}(x)$. Thus, $\tilde{g}$ is onto. So, $|A| = |g(A)|$. The proof is complete.

   Now, $f(S) \subseteq \mathbf{N}$ contains the infinite set $\{f(n, 1) : n \in \mathbf{N}, n \geq 2\} = \{2^n 3 : n = 2, 3, \ldots\}$. Thus, $f(S)$ cannot be empty or finite. So, $f(S)$ is denumerable, and $|S| = |f(S)| = |\mathbf{N}|$.

3. For $k \in \mathbf{N}$, let $S_k = \{A \subseteq \mathbf{N} : |A| = k\}$. Show that $|S_2| = |\mathbf{N}|$

   Solution. Define $f : S_2 \to \mathbf{N}$ by $f(\{a, b\}) = 2^u 3^v$ with $u = \min\{a, b\}, v = \max\{a, b\}$. Then for any $\{a, b\} \in S_2$, we have a unique image $2^u 3^v \in \mathbf{N}$. Hence, f is a well-defined function. Suppose $f(\{a, b\}) = 2^u 3^v$ and $f(\{\alpha, \beta\}) = 2^r 3^s$. Then $(u, v) = (r, s)$ so that the set $\{a, b\}$ and $\{\alpha, \beta\}$ have the same maximum and minimum. Hence, $\{a, b\} = \{\alpha, \beta\}$. So, f is one-one. Thus, $|S_2| = |f(S_2)|$ by the previous lemma. Now, $f(S_2) \subseteq \mathbf{N}$ contains the infinite set

   $$\{f(\{1, n\}) : n \in \mathbf{N} - \{1\}\} = \{2 \cdot 3^n : n = 2, 3, \ldots\}.$$

   Thus,$f(S_2)$ is dunumerable, and $|S_2| = |f(S_2)| = |\mathbf{N}|$.

4. (Extra 4 points) Using the definition of S_k from problem 3, show that

   (a) for all $k \in \mathbf{N}$, S_k is denumerable.

   (b) $\mathcal{S} = \bigcup_{k=1}^{\infty} S_k$ is denumerable.

   Solution. (a) For any $\{a_1, \ldots, a_k\} \in S_k$, we may assume that $a_1 \leq \cdots \leq a_k$. Let $P = \{p_1, p_2, \ldots\}$ be the denumerable set of prime numbers. Define $f : S_k \to \mathbf{N}$ by

   $$f(\{a_1, \ldots, a_k\}) = p_1^{a_1} \cdots p_k^{a_k}.$$

   Then one can check that f is a well-defined injection, and $|S_k| = |f(S_k)| = |\mathbf{N}|$ by an argument similar to that in the proof of $|S_2| = |\mathbf{N}|$.

   (b) Since S_k is denumerable, we may write $S_k = \{A_{k1}, A_{k2}, A_{k3}, \ldots\}$ for $k = 1, 2, \ldots$. Now, define $f : \cup_{k\in\mathbf{N}} S_k \to \mathbf{N} \times \mathbf{N}$ by $f(A_{ij}) = (i, j)$. Then one readily checks that f is a bijection so that $| \cup_{k\in\mathbf{N}} S_k| = |\mathbf{N} \times \mathbf{N}| = |\mathbf{N}|$.

5. Let $\emptyset \neq J \subseteq \mathbf{N}$. For each $j \in J$, A_j is a non-empty countable set. Show that $\cup_{j\in J} A_i$ is countable.

   Solution. Case 1. Suppose $J = \{j_1, \ldots, j_m\}$ is finite. Let $A_{j_k} = \{a_{k1}, a_{k2}, \ldots\}$, which may be finite or denumerable, and let $A = \cup_{j\in J} A_j$. Define $f : A \to \mathbf{N}$ by $f(a_{1m}) = 2^1 3^m$ if $a_{1m} \in A_{j_1}$; for $k \geq 2$, define $f(a_{km}) = 2^k 3^m$ if $a_{km} \in A_{j_k} - (\cup_{1\leq\ell<k} A_{j_\ell})$. Note that this will avoid sending a_{km} to a different element in $\mathbf{N}$ if it already appears in A_{j_ℓ} for some $\ell < k$.

   Now, f is well-defined because every $a_{ij} \in \cup_{j\in J} A_j$ has an image in $\mathbf{N}$, and it will be assigned to different values. Also, f is an injection because $2^i 3^j = f(a_{ij}) = f(a_{rs}) = 2^r 3^s$ will imply that $(i, j) = (r, s)$ so that $a_{ij} = a_{rs}$. Thus, $|A| = |f(A)|$. Now, A is non-empty so that $f(A)$ is non-empty. Thus, $f(A)$ is finite or denumerable, i.e., $f(A)$ is countable and so is A.

   Case 2. Suppose $J = \{j_1, j_2, \ldots\}$ is denumerable. We can use the same argument as in Case 1 and construct an injection $f : A \to \mathbf{N}$ and conclude that A is countable.

6. Let $A = \{(\alpha_1, \alpha_2, \alpha_3, \ldots) : \alpha_i \in \{0, 1\}, i \in \mathbf{N}\}$. Show that there is no surjection $f : \mathbf{N} \to A$.

   Solution. Suppose $f : \mathbf{N} \to A$ is a sujection. Let $f(k) = (a_{k1}, a_{k_2}, a_{k3}, \ldots)$ for $k \in \mathbf{N}$. Consider $b = (b_1, b_2, \ldots)$ such that $b_j \neq a_{jj}$ for each $j \in \mathbf{N}$. Then $b \neq f(k)$ for any k because b_k differs from a_{kk}. This is contradiction.

7. (6 points) Determine the cardinality of the following sets (finite, denumerable, or uncountable), and justify your answers:

   (a) the set of all open intervals with rational midpoints.

   (b) the set of all open intervals with rational endpoints.

   Solution. (a) The set is uncountable because it contains the set S of intervals of the form $I_r = (-r, r)$ with $r \in (0, 1)$. Clearly, $f : (0, 1) \to S$ defined by $f(r) = I_r$ is a bijection. So, $|S| = |(0, 1)|$, and hence S is uncountable. As a result, the set of all intervals with rational midpoints is uncountable.

   (b) The set $R = \{(a, b) : a, b \in \mathbf{Q}, a < b\}$ is denumerable. We can construct $f : R \to \mathbf{Q} \times \mathbf{Q}$ such that $f((a, b)) = (a, b)$, i.e., f send the interval (a, b) to the ordered pair (a, b). Then f is one-one so that $|R| = |f(R)|$. Now, $|\mathbf{Q}| = |\mathbf{N}|$ implies that $|\mathbf{Q} \times \mathbf{Q}| = |\mathbf{N} \times \mathbf{N}|$ a result proved in class. So, $f(R)$ is a subset of a denumerable set containing all the pairs $(0, n)$ for $n \in \mathbf{N}$. Therefore, $f(R)$ is not finite, and we have $|R| = |f(R)| = |\mathbf{N}|$.

8. Show that $f : (0,1) \times (0,1) \to (0,1)$ defined by

$$f(0.a_1a_2\cdots, 0.b_1b_2\cdots) = 0.a_1b_1a_2b_2a_3b_3\cdots$$

is bijective.

Solution. If $f(0.a_1a_2\cdots, 0.b_1b_2\cdots) = f(0.c_1c_2\cdots, 0.d_1d_2\cdots)$, then $0.a_1b_1a_2b_2\cdots = 0.c_1d_1c_2d_2\cdots$. Hence, $0.a_1a_2\cdots, 0.b_1b_2\cdots = 0.c_1c_2\cdots, 0.d_1d_2\cdots$. So, f is one-one.

for any $d = 0.d_1d_2\cdots \in (0,1)$, we have $f(0.d_1d_3d_5\cdots, 0.d_2d_4d_4\cdots) = d$. Hence f is onto.

Combining the above we see that f is a bijection.

9. Let $A = (0,1) \cup (2,3)$ and $B = [1,2]$. Construct an injection from A to B, and construct an injection from B to A.

Solution. Define $f : A \to B$ by $f(x) = 1 + x/3$. Then for any $x \in A$, $0 < x < 3$ so that $1 < f(x) < 2$. Hence $f(x) \in B$. Clearly, if $f(a) = f(b)$, then $1 + a/3 = 1 + b/3$ so that $a = b$. Hence, f is one-one.

Define $g : B \to A$ by $g(x) = x/3$. Then for any $x \in B$, $1/3 < g(x) < 2/3$. Thus, $g(x) \in A$. Clearly, $g(a) = g(b)$ implies $a/3 = b/3$ so that $a = b$. Hence, g is one-one.

Remark We will be able to conclude that there is a bijection between the two sets by a future (Schröder-Bernstein) theorem. So, you do not need to construct the bijection!