Math 214 Homework 10 Your name

Four points for each problem unless specified otherwise.

1. Show that if A and B are denumerable sets, then $A \cup B$ is also denumerable

   [Hint: Let $A = \{a_1, a_2, \ldots\}$, and let $C = B - A$. Then $A \cup B = A \cup C$. Consider three cases: $C = \emptyset$, C is finite, C is denumerable.]

2. Prove that $S = \{(a, b) : a, b \in \mathbf{N}, a \geq 2b\}$ is denumerable.

   [Hint: Show that $f : S \to \mathbf{N}$ defined by $f(a, b) = 2^a 3^b$ is a one-one function, and deduce that $|S| = |f(S)|$ is denumerable. You need to argue that $f(S)$ is not finite.]

3. For $k \in \mathbf{N}$, let $S_k = \{A \subseteq \mathbf{N} : |A| = k\}$. Show that $|S_2| = |\mathbf{N}|$

   [Hint: Define $f : S_2 \to \mathbf{N}$ by $f(\{a, b\}) = 2^u 3^v$ with $u = \min\{a, b\}, v = \max\{a, b\}$. Then argue that $|S_2| = |f(S_2)|$ and $f(S_2)$ is denumerable as in the last question.]

4. (Extra 4 points) Using the definition of S_k from problem 3, show that

   (a) for all $k \in \mathbf{N}$, S_k is denumerable.

   (b) $\mathcal{S} = \bigcup_{k=1}^{\infty} S_k$ is denumerable.

5. Let $\emptyset \neq J \subseteq \mathbf{N}$. For each $j \in J$, A_j is a non-empty countable set. Show that $\cup_{j \in J} A_i$ is countable.

   [Hint: Let $J = \{j_1, j_2, \ldots\}$ that may be finite or denumerable. Let $A_{j_\ell} = \{a_{\ell 1}, a_{\ell 2}, \ldots, \}$ be nonempty countable (that may be finite1 or infinite) . Consider the list of primes $p_1, p_2, \ldots$ arranged in ascending order. Define $f : \cup_{j \in J} A_j \to \mathbf{N}$ by $f(a_{1,k}) = p_1^k$, and for $i > 1$ $f(a_{i,k}) = p_i^k$ if $a_{i,k} \notin \cup_{\ell \leq i} A_{j_\ell}$. Show that f is injective and hence $|\cup_{j \in J} A_j| = |B|$ with $B = \{f(x) : x \in \cup_{j \in J} A_j\}$. Then derive the conclusion.]

6. Let $A = \{(\alpha_1, \alpha_2, \alpha_3, \ldots) : \alpha_i \in \{0, 1\}, i \in \mathbf{N}\}$. Show that there is no surjection $f : \mathbf{N} \to A$.

7. (6 points) Determine the cardinality of the following sets (finite, denumerable, or uncountable), and justify your answers:

   (a) the set of all open intervals with rational midpoints.

   (b) the set of all open intervals with rational endpoints.

8. Show that $f : (0, 1) \times (0, 1) \to (0, 1)$ defined by

   $$f(0.a_1 a_2 \cdots, 0.b_1 b_2 \cdots) = 0.a_1 b_1 a_2 b_2 a_3 b_3 \cdots$$

   is bijective.

9. Let $A = (0, 1) \cup (2, 3)$ and $B = [1, 2]$. Construct an injection from A to B, and construct an injection from B to A.