Math 214 – Foundations of Mathematics   Homework 11   Sample Solution

1. (2 points for each part.) Determine with proofs the following sets are finite, denumerable or uncountable:
(a) $\{(x,y) : x, y \in \mathbf{N}, \text{ and } x+y = 10000\}$.
(b) All the functions from $\mathbf{Z}_3$ to $\mathbf{R}$.
(c) The set of all functions from $\mathbf{N}$ to $\{0,1\}$.

   Solution. (a) The set $\{(x, 10000-x) : 1 \le x \le 9999\}$ has 9999 elements and is finite.

   (b) The given set is uncountable. Reason. Consider the subset S_1 of the giving set consisting of functions $f_r : \mathbf{Z}_3 \to \mathbf{R}$ defined by $f_r([0]) = f_r([1]) = f_r([2])) = r$ for every $r \in R$. One can check that $\phi : \mathbf{R} \to S_1$ defined by $\phi(r) = f_r \in S_1$ is a bijection. Thus, S_1 is uncountable and so is the given set.

   (c) Every function $f : \mathbf{N} \to \{0,1\}$ can be identified with a sequence $(f(1), f(2), \cdots)$, where each term lies in the set $\{0,1\}$. This is the set $2^{\mathbf{N}}$, which is known that the set is uncountable by the previous homework.

   Alternatively, the set is $2^{\mathbf{N}}$ which has the same cardinality of $\mathbf{R}$ and is uncountable.

2. (2 points for each part.) Let A_1, A_2, B_1, B_2 be non-empty sets such that $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 = \emptyset$. Suppose $f_1 : A_1 \to B_1$, $f_2 : A_2 \to B_2$ are functions. Define $f : A_1 \cup A_2 \to B_1 \cup B_2$ by

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A_1, \\ f_2(x) & \text{if } x \in A_2. \end{cases}$$

   (a) Show that f is a well-defined function.

   (b) If f_1, f_2 are injective, show that f is injective.

   (c) If f_1, f_2 are surjective, show that f is surjective.

   Solution. (a) For any $a \in A_1 \cup A_2$, if $a_1 \in A_1$, there is a unique image $f_1(a_1) \in B_1$; if $a \in A_2$, there is a unique image $f_2(a_2) \in B_2$. Thus, $f(a)$ is uniquely determined in $B_1 \cup B_2$.

   (b) Suppose f_1, f_2 are injective. Suppose $x_1 \ne x_2$ in $A_1 \cup A_2$. If $x_1, x_2 \in A_1$ or $x_1, x_2 \in A_2$, then $f(x_1) \ne f(x_2)$ by the injectiveness of f_1 and f_2. If x_1, x_2 are not in the same sets A_1, A_2, the $f(x_1), f(x_2)$ will not be in the same sets B_1, B_2. So, they are different.

   (c) Suppose f_1, f_2 are surjective. Suppose $b \in B_1 \cup B_2$. By the surjectiveness of f_1 and f_2, if $b \in B_1$, there is $a \in A_1$ such that $f(a) = f_1(a) = b$; if $b \in B_2$, then there is $a \in A_2$ such that $f(a) = f_2(a) = b$.

3. Let $A = \mathbf{N} - \{n^2 : n \in \mathbf{N}\}$. Construct a bijection from A to $\mathbf{N}$. Verify your answer.

   Solution. Define $f : A \to \mathbf{N}$ by $f(x) = x - n$ if $n^2 < x < (n+1)^2$. Clearly, $f(x) = x - n > n^2 - n \ge 0$ is a positive integer for every $x \in A$. Thus, f is well-defined.

   Injective: Suppose $x < y$ in A such that $p^2 < x < (p+1)^2$, $q^2 < y < (q+1)^2$.

   If $p = q$, then $f(x) = x - p < y - q = f(y)$.

   If $p < q$, then $f(y) = y - q > q^2 - q$ and $f(x) = x - p < (p+1)^2 - p$ so that $f(x) \le (p+1)^2 - (p+1)$ and $f(y) - f(x) > [q^2 - q] - [(p+1)^2 - (p+1)] = (q-p-1)(q+p) \ge 0$ so that $f(y) > f(x)$.

   Surjective: For every $y \in \mathbf{N}$, we want to choose $x \in A$ such that $p^2 < x < (p+1)^2$ so that $f(x) = x - p = y$. Thus, we want to find p such that $x = y + p$ and $p^2 < y + p < (p+1)^2$. So, we should determine $p \in \mathbf{N}$ such that $p^2 - p < y < (p+1)^2 - p$. This is always possible because the set $\{n : y \le (p+1)^2 - (p+1)\}$ is non-empty and has a smallest number by the well-ordering principle. Thus, one can always find a unique $p \in \mathbf{N}$ such that $p^2 - p < y \le (p+1)^2 - (p+1)$. Then $x = y + p$ will satisfy $f(x) = y$.

4. Let $A = \{a_1, a_2, \ldots\}$ be a denumerable set.

   (a) (4 point) Prove that for every $n \in \mathbf{N}$, A can be partitioned into n denumerable sets.

   (b) (4 point) Prove that A can be partitioned into infinitely many denumerable sets.

   Solution. (a) Suppose $n \in \mathbf{N}$. For $k = 1, \ldots, n$, let

   $$A_k = \{a_j : j = (m-1)n + k, m \in \mathbf{N}\} = \{a_k, a_{n+k}, a_{2n+k}, \ldots\}.$$

   Then each A_k is a denumerable set and A is a disjoint union of $A_1, \ldots, A_n$.

   (b) For every prime p, let $B_p = \{a_j : j = p^m, m \in \mathbf{N}\}$, which is clearly denumerable, and let $B_0 = \{a_j : j \ne p^m$ for any prime p and any $m \in \mathbf{N}\} \subseteq A$. Then B_0 contains the subset $T = \{a_j : j = 2^m 3, m \in \mathbf{N}\}$. So, B_0 is not finite, and is a denumerable subset of A. Thus, A is a disjoint union of the denumerable sets $A_0, A_2, A_3, \ldots$.

5. (2 points for each part.) Suppose $A \subseteq B$. Prove or disprove the following.

   (a) If B is denumerable, then A is denumerable. (a) False. $A = \{1\}, B = \mathbf{N}$.

   (b) If A is denumerable, then B is denumerable. (b) False. $A = \mathbf{N}, B = \mathbf{R}$.

   (c) If B is uncountable, then A is uncountable. (c) False. $A\{1\}, B = \mathbf{R}$.

   (d) If A is uncountable, then B is uncountable. (d) True. Suppose A is uncountable. If B is countable, then A will be countable, which is a contradiction.

6. Prove that the set of infinite subsets of $\mathbf{N}$ is uncountable.

   Solution. Note that $\mathcal{P}(\mathbf{N})$ can be partitioned into $X \cup Y$, where X is the set of all finite subsets of $\mathbf{N}$ and Y is the set of all infinite subsets of $\mathbf{N}$. By a previous homework, $X = \cup_{n\in\mathbf{N}} S_n$, where S_n is the set of subsets of $\mathbf{N}$ with $\mathbf{N}$ elements, and X is countable. If Y is also countable, then $X \cup Y = \mathcal{P}(\mathbf{N})$ is countable, which contradicts the fact that $\mathcal{P}(\mathbf{N})$ is uncountable. So, Y must be uncountable.

7. (4 points) Show that $|\mathbf{N}^n| = |\mathbf{N}|$ for any $n \in \mathbf{N}$.

   Solution. We prove the result by induction. If $n = 1$, the result is known. Suppose the result holds for $k \ge 1$, i.e., $|\mathbf{N}^k| = |\mathbf{N}|$. Then $\mathbf{N}^{k+1} = \mathbf{N}^k \times N$ is the product set of two denumerable set so that $\mathbf{N}^{k+1}$ is denumerable. By the principle of MI, the result follows.

   Alternatively, consider the function $f : \mathbf{N}^n \to \mathbf{N}$ defined by $f(a_1, \ldots, a_n) = p_1^{a_1} \cdots p_n^{a_n} \in \mathbf{N}$, where $p_1, \ldots, p_n$ are distinct primes. Then f is an injection and $f(\mathbf{N}^n) = T \subseteq$ is denumerable so that $|\mathbf{N}^n| = |T| = |\mathbf{N}|$.

8. Give an example of a set A with subsets B and C such that

   $$|\mathbf{N}| < |C| < |B| < |A|.$$

   Solution. Let $A = \mathbf{R} \cup \mathcal{P}(\mathbf{R}) \cup \mathcal{P}(\mathcal{P}(\mathbf{R}))$, $B = \mathbf{R} \cup \mathcal{P}(\mathbf{R})$, $C = \mathbf{R}$.

   Then $|C| = |\mathbf{R}|$, $B = |\mathcal{P}(\mathbf{R}))|$, and $|A| \ge |\mathcal{P}(\mathcal{P}(\mathbf{R}))|$.